EFFECTIVE HAMILTONIANS FOR ATOMS IN VERY STRONG MAGNETIC FIELDS

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Abstract. We propose three effective Hamiltonians which approximate atoms in very strong homogeneous magnetic fields B modelled by the Pauli Hamiltonian, with fixed total angular momentum with respect to magnetic field axis. All three Hamiltonians describe N electrons and a fixed nucleus where the Coulomb interaction has been replaced by B-dependent one-dimensional effective (vector valued) potentials but without magnetic field. Two of them are solvable in at least the one electron case. We briefly sketch how these Hamiltonians can be used to analyse the bottom of the spectrum of such atoms.

1. Introduction

The Pauli Hamiltonian of a non-relativistic atom with an infinitely heavy nucleus and electrons with spin in a constant magnetic field $\mathbb{B}$ of strength B is given by:

$$H^B(Z,N) = \sum_{j=1}^{N} \left(\frac{1}{2}(\frac{1}{i}\nabla_j - \frac{1}{2}\mathbb{B}\wedge r_j)^2 + \vec{\sigma}_j \cdot \mathbb{B} - \frac{Z}{|r_j|} \right) + \sum_{1\le j<k\le N} \frac{1}{|r_j - r_k|}, \tag{1}$$

where $r_j = (x_j, y_j, z_j) \in \mathbb{R}^3$ are the coordinates of the j-th electron, $\vec{\sigma}_j$ is its spin, and ∇_j is the gradient with respect to r_j. Note that we have made the choice of $\frac{1}{2}\mathbb{B}\wedge r$ for the vector potential of $\mathbb{B}$, and that we are working in atomic units.

We fix the direction of $\mathbb{B}$ to be the z-direction: $\mathbb{B} = B(0,0,1)$ with $B \ge 0$ without loss of generality. Recall that the z-component of $\vec{\sigma}_j$ is given, in the Pauli representation, by

$$I \otimes \cdots \sigma_{z_j} \cdots \otimes I, \quad \sigma_{z_j} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

acting on the N-fold tensor product $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$. It is known (see [KaKu]) that $H^B := H^B(Z,N)$ defines an essentially self-adjoint operator on

$$\mathcal{H} = \bigotimes_{j=1}^{N} L^2(\mathbb{R}^3) \otimes \mathbb{C}^2, \tag{2}$$

the Hilbert space of distinguishable electrons or "boltzons" (meaning particles satisfying the Boltzman statistics: we thank Beth Ruskai for introducing us to this expression). Physical atoms are of course modeled by H^B restricted to the fermionic subspace

$$\mathcal{H}_{\mathrm{f}} = \bigwedge_{j=1}^{N} L^2(\mathbb{R}^3) \otimes \mathbb{C}^2, \tag{3}$$

of totally anti-symmetric wave-functions in $\mathcal{H}$, where $\wedge$ stands for exterior product. A useful alternative description of $\mathcal{H}$, used in atomic physics, is given by the unitary map

$$\mathcal{H} \to L^2\left((\mathbb{R}^3 \times \{\pm 1\})^N\right), \ \psi \to U(\psi)(r_1, s_1; \cdots; r_N, s_N),$$

the isometry being defined by taking the components of $\psi(r_1, \cdots, r_N)$ with respect to the natural basis of $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ consisting of the products $\otimes_j e_{s_j}$ of the normalized eigenvectors $e_\pm$ of σ_{z_j}, e_- corresponding to "spin down" and e_+ to "spin up". In this new representation, σ_{z_j} acts as the multiplication operator by $s_j/2$, and the fermionic subspace $\mathcal{H}_{\mathrm{f}}$ of (2) is simply obtained by anti-symmetrizing with respect to the 4-tuples of variables $(r_1, s_1), \cdots, (r_N, s_N)$.

The main results of this paper can be summarized in the following five theorems below. It is worthwhile to observe that H^B commutes with each individual spin operator σ_{z_j} and therefore decomposes in a direct sum which is unitarily equivalent to

$$\bigoplus_{s_{z_j} \in \{\pm 1\}} H^{B, \mathbb{S}_z = -\frac{NB}{2}} + \sum_{j=1}^{N} (1 + s_{z_j}) \frac{B}{2},$$

where $\mathbb{S}_z := \sum_{j=1}^N \sigma_{z_j}$ denotes the z component of the total spin operator. So from now on we will consider only $H^{B, \mathbb{S}_z = -\frac{NB}{2}}$ and denote it again by H^B. Notice that this operator simply acts in $\otimes_{j=1}^N L^2(\mathbb{R}^3)$. Since the Hamiltonian H^B also commutes with the total angular momentum operator in the field direction, which we will call $\mathbb{L}_z$, we can fix a value $\mathbb{M} \geq 0$ of the latter. Our results will imply that the bottom of the spectrum of H^B will necessarily occur for a non-negative value of $\mathbb{M}$, and we will therefore restrict ourselves to $\mathbb{M} \geq 0$. Let $H^{B,\mathbb{M}}$ be the restriction of H^B to the $\mathbb{M}$-th angular momentum channel (in the field direction), and let $\Pi_{\mathrm{eff}}^{B,\mathbb{M}}$ be the orthogonal projection onto the lowest Landau states with z-angular momentum $\mathbb{M}$ (cf. (31) for the precise definition). We define the *effective* Hamiltonian $h_{\mathrm{eff}} = h_{\mathrm{eff}}^{B,\mathbb{M}}$ by

$$h_{\mathrm{eff}}^{B,\mathbb{M}} = \Pi_{\mathrm{eff}}^{B,\mathbb{M}} \, H^{B,\mathbb{M}} \, \Pi_{\mathrm{eff}}^{B,\mathbb{M}}, \tag{4}$$

and let

$$\Pi_{\perp}^{B,\mathbb{M}} = I - \Pi_{\mathrm{eff}}^{B,\mathbb{M}}$$

be the projection onto the orthogonal complement (always restricting ourselves to the $\mathbb{M}$-th z-angular momentum channel). The operator $h_{\mathrm{eff}}^{B,\mathbb{M}}$ is the first, and most encompassing, of three "effective Hamiltonians" we will consider in this paper. The two other ones, called $h_C^{B,\mathbb{M}}$ and h_δ^B, will be defined below. It will be convenient to complete $h_{\mathrm{eff}}^{B,\mathbb{M}}$ as follows

$$H_{\mathrm{eff}}^{B,\mathbb{M}} := h_{\mathrm{eff}}^{B,\mathbb{M}} \oplus H_{\perp}^{B,\mathbb{M}}, \quad \text{with} \quad H_{\perp}^{B,\mathbb{M}} := \Pi_{\perp}^{B,\mathbb{M}} H^{B,\mathbb{M}} \Pi_{\perp}^{B,\mathbb{M}}. \tag{5}$$

For any self-adjoint operator A, we let $\sigma(A)$ denote the spectrum of A and $\rho(A)$ its resolvent set. Our first main result is:

Theorem 1.1. *Let $\alpha = \alpha(B)$ be the unique positive solution of the equation*

$$\alpha + \log \alpha = \frac{1}{2} \log B, \tag{6}$$

and let $d_{\mathrm{eff}}(\xi) = \mathrm{dist}\left(\xi, \sigma(h_{\mathrm{eff}}^{B,\mathbb{M}})\right)$. There exist positive constants B_{eff}, c_{eff} and C_{eff}, which only depend on Z, N and $\mathbb{M}$, such that for all $B \geq B_{\mathrm{eff}}$, and all real ξ satisfying

$$c_{\mathrm{eff}} \frac{\alpha}{\sqrt{B}} \leq d_{\mathrm{eff}}(\xi) \leq \frac{1}{2} \alpha^2.$$

we have that $\xi \in \rho(H^{B,\mathbb{M}})$, and

$$\| \, (H^{B,\mathbb{M}} - \xi)^{-1} - (H_{\mathrm{eff}}^{B,\mathbb{M}} - \xi)^{-1} | \leq C_{\mathrm{eff}} \frac{\alpha(B)^2}{d_{\mathrm{eff}}(\xi)^2 \, \sqrt{B}}. \tag{7}$$

Remarks 1.2. (i) We shall see in Theorem 3.1 that $\sigma(H_\perp^{B,\mathbb{M}}) \subset \mathbb{R}_+$ for $B \geq B_{(36)}$, where the latter is defined by formula (36)[1]. Since $B_{\text{eff}} \geq B_{(36)}$ by (107), and since one can see from (9) and (10) below that $\mathbb{R}_+ \subset \sigma(h_{\text{eff}}^{B,\mathbb{M}})$, it follows that $\sigma(h_{\text{eff}}^{B,\mathbb{M}}) = \sigma(H_{\text{eff}}^{B,\mathbb{M}})$ when $B \geq B_{\text{eff}}$.

(ii) The equation for $\alpha(B)$ is equivalent to $\alpha e^\alpha = \sqrt{B}$, and can therefore be written as

$$\alpha(B) = W(\sqrt{B}),$$

where W is the principal branch of the Lambert W function, see e.g. [CoGoHaJeKn]. Using known properties of the Lambert W-function, or by elementary arguments, one shows that

$$\alpha(B) = \frac{1}{2}\log B - \log^{(2)} B + \log 2 + O\left(\frac{\log^{(2)} B}{\log B}\right), \quad B \to \infty, \tag{8}$$

where $\log^{(2)}(x) := \log(\log x)$, for $x > 1$. In particular, $\alpha(B) \simeq \log(\sqrt{B})$, as $B \to \infty$.

(iii) Our proof yields explicit constants B_{eff}, c_{eff} and C_{eff}, for given $N, \mathbb{M}$ and Z. This is also true for the constants in Theorems 1.3 and 1.5 below.

(iv) The upper bound $\alpha^2/2$ on $d_{\text{eff}}(\xi)$ is only there to allow a simple expression for the upper bound in (7), and is by no means essential. The same remark applies to Theorems 1.3, 1.5 and 1.6 below.

Some applications of Theorem 1.1, as well as of Theorems 1.3, 1.5, 1.6 and 1.8 below, to the study of spectral properties of $H^{B,\mathbb{M}}$ are given in the concluding remarks section.

The operator $h_{\text{eff}}^{B,\mathbb{M}}$ has the structure of a multi-particle Schrödinger operator on the real line, $\mathbb{R}$:

$$h_{\text{eff}}^{B,\mathbb{M}} = -\frac{1}{2}\Delta - Z\sum_j V_j^{B,\mathbb{M}}(z_j) + \sum_{j<k} V_{jk}^{B,\mathbb{M}}(z_j - z_k), \tag{9}$$

with operator-valued potentials acting point-wise on a certain finite-dimensional Hilbert-space $F_{\mathbb{M}}^B$, defined in (32) below. Essentially, $F_{\mathbb{M}}^B$ is the vector space spanned by the lowest Landau states with angular momentum $\mathbb{L}_z = \mathbb{M}$. As we will see, Ran $\Pi_{\text{eff}}^{B,\mathbb{M}}$ is canonically isomorphic to the space $L^2(\mathbb{R}^N, F_{\mathbb{M}}^B)$ of vector-valued L^2-functions , and the potentials in (9) are simply obtained by projecting the respective Coulomb terms in (1) along $\Pi_{\text{eff}}^{B,\mathbb{M}}$:

$$V_j^{B,\mathbb{M}}(z_j) := \Pi_{\text{eff}}^{B,\mathbb{M}}\frac{1}{|r_j|}\Pi_{\text{eff}}^{B,\mathbb{M}} \qquad V_{jk}^{B,\mathbb{M}}(z_j - z_k) := \Pi_{\text{eff}}^{B,\mathbb{M}}\frac{1}{|r_j - r_k|}\Pi_{\text{eff}}^{B,\mathbb{M}}. \tag{10}$$

We will show that the potentials (10) can be approximated by certain simpler ones, which will give rise to our two other effective Hamiltonians. Define the tempered distribution q^B on $\mathbb{R}$ by:

$$q^B(z) = \log B\ \delta(z) + \mathrm{Pf}\ \left(\frac{1}{|z|}\right), \tag{11}$$

where

$$\mathrm{Pf}\left(\frac{1}{|z|}\right) := \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{sgn}(z)\log|z|\right),$$

(with distributional derivative) is the finite part (in the sense of Hadamard) of the singular function $1/|z|$; $\mathrm{Pf}(|z|^{-1})$ should be interpreted as a regularization of the (3-dimensional) Coulomb potential restricted to the line. Also introduce (constant)

[1]see our convention on constants at the end of this introduction

finite dimensional operators $C_j^{n,B,\mathbb{M}}$ and $C_{jk}^{e,B,\mathbb{M}}$, acting on the vector space $F_{\mathbb{M}}^B$ introduced above, defined by

$$C_j^{n,B,\mathbb{M}} := -\Pi_{\mathrm{eff}}^{B,\mathbb{M}} \log\left(\frac{B}{4}(x_j^2 + y_j^2)\right)\Pi_{\mathrm{eff}}^{B,\mathbb{M}}, \tag{12}$$

and

$$C_{jk}^{e,B,\mathbb{M}} := -\Pi_{\mathrm{eff}}^{B,\mathbb{M}} \log\left(\frac{B}{4}\left((x_j - x_k)^2 + (y_j - y_k)^2\right)\right)\Pi_{\mathrm{eff}}^{B,\mathbb{M}}. \tag{13}$$

The superscripts "n" and "e" stand for "nucleus" and "electron", respectively, as a reminder that (12) is a vestige of the interaction between the j-th electron and the nucleus, while (13) originates in the electron-electron interaction between electrons j and k. Finally, define an operator $h_C^{B,\mathbb{M}}$ on $L^2(\mathbb{R}, F_{\mathbb{M}}^B)$ (the 'C' standing for 'Coulomb') by

$$\begin{aligned} h_C^{B,\mathbb{M}} \quad = \quad & -\frac{1}{2}\Delta - Z\sum_j \left(q^B(z_j) + C_j^{n,B,\mathbb{M}}\delta(z_j)\right) \\ & + \sum_{j<k}\left(q^B(z_j - z_k) + C_{jk}^{e,B,\mathbb{M}}\delta(z_j - z_k)\right). \end{aligned} \tag{14}$$

As we will see in section 4, the right hand side of (14) defines a self-adjoint operator $h_C^{B,\mathbb{M}}$ on $L^2(\mathbb{R}^N)$, despite the distributional potentials. This will be a consequence of the Kato-Lax-Lions-Milgram-Nelson Theorem. The form domain of $h_C^{B,\mathbb{M}}$ is simply the vector-valued first Sobolev space $H^1(\mathbb{R}^N; F_{\mathbb{M}})$, while its operator domain will be characterized in appendix A. As in (5) we introduce

$$H_C^{B,\mathbb{M}} := h_C^{B,\mathbb{M}} \oplus H_\perp^{B,\mathbb{M}}.$$

Our second main theorem then is the following:

Theorem 1.3. *Let $\alpha = \alpha(B)$ be as in Theorem 1.1, and put $d_C(\xi) := \mathrm{dist}\left(\xi, \sigma(h_C^{B,\mathbb{M}})\right)$. There exists positive constants B_C, c_C and C_C which depend only on Z, N and $\mathbb{M}$, such that for all $B \geq B_C$ and all real ξ satisfying*

$$c_C\frac{\alpha^{3/2}}{B^{1/4}} \leq d_C(\xi) \leq \frac{1}{4}\alpha^2.$$

we have that $\xi \in \rho(H^{B,\mathbb{M}})$, and

$$\|(H^{B,\mathbb{M}} - \xi)^{-1} - (H_C^{B,\mathbb{M}} - \xi)^{-1}\| \leq \frac{C_C\alpha^{\frac{3}{2}}}{B^{1/4}d_C(\xi)^2}. \tag{15}$$

Remark 1.4. In top order in B, all that remains of the electrostatic potentials in $H^{B,\mathbb{M}}$ are the extremely short-range δ-potentials. In next order, the long range character of the original Coulomb potentials reasserts itself in two ways: in the magnetic field direction, through the Pf $(|\cdot|^{-1})$-terms in $h_C^{B,\mathbb{M}}$, and in the transversal directions, through the $C_j^{n,B,\mathbb{M}}$- and $C_{jk}^{e,B,\mathbb{M}}$- terms. The latter are in fact simply the quantum mechanical mean, with respect to the projection onto the lowest Landau band states of total angular momentum $\mathbb{M}$ (in the field direction), of a 2-dimensional logarithmic potential, minus a B-dependent constant. This logarithmic potential is the natural electrostatic potential for the plane. Physically, this can be understood as follows: under the influence of the strong magnetic field the electrons will spiral closely around the field lines, along circles of radius $O(B^{-1/2})$ in the plane transversal to the field, while occupying an interval of size $O((\log B)^{-1})$ in the field direction itself, as a consequence of the nuclear attraction. For big B, and at different locations in the (x, y)-plane, they will see each other and the

nucleus as so many infinitely long charged wires, and, as is known from classical electrostatics, such wires interact via a logarithmic potential.

A simpler effective Hamiltonian, our third and last one, and historically the first to be proposed (cf. [LSY], [BaSoY], [BD]), is roughly speaking obtained by only keeping the leading term in the potential of $h_C^{B,\mathbb{M}}$. More precisely, we put:

$$h_\delta^B = -\frac{1}{2}\Delta_z + 2\alpha(B)\, v_\delta, \tag{16}$$

where

$$v_\delta(z) = -Z\sum_{j=1}^N \delta(z_j) + \sum_{j<k}\delta(z_j - z_k). \tag{17}$$

Looking back at (11) it would seem natural to take as potential $\log B v_\delta$, but it turns out that $2\alpha(B)v_\delta$ leads to smaller error estimates; notice that in view of (8), $2\alpha v_\delta$ is also a $O(\log B)$ part of the potential in h_C^B. Furthermore, with this choice the coupling constant $2\alpha(B)$ is positive for all $B>0$ which is not the case for $\log B$. Contrary to our previous two effective Hamiltonians, h_δ^B does not explicitly depend on $\mathbb{M}$ anymore, but it will operate on an $\mathbb{M}$ and B-dependent Hilbert space, namely $L^2(\mathbb{R}^N, F_{\mathbb{M}}^B) \simeq L^2(\mathbb{R}^N)\otimes F_{\mathbb{M}}^B$ (which in fact are canonically isomorphic for different B). Considering (16) as acting on scalar $L^2(\mathbb{R}^N)$, we define the δ-model as being the operator

$$h_\delta^{B,\mathbb{M}} := h_\delta^B \otimes I_{F_{\mathbb{M}}^B}, \qquad I_{F_{\mathbb{M}}^B} \text{ being the identity operator.} \tag{18}$$

We will often simply write h_δ^B, except when we want to stress the vector-valued nature of the L^2-functions in the domain. Again as in (5) we introduce

$$H_\delta^{B,\mathbb{M}} := h_\delta^{B,\mathbb{M}} \oplus H_\perp^{B,\mathbb{M}}.$$

Our third approximation theorem is:

Theorem 1.5. *Let $\alpha := \alpha(B)$ be as in Theorem 1.1, and put $d_\delta(\xi) := \mathrm{dist}\left(\xi, \sigma(h_\delta^{B,\mathbb{M}})\right)$. There exist positive constants B_δ, c_δ and C_δ, depending on N, Z and $\mathbb{M}$, such that for all $B \geq B_\delta$ and real ξ satisfying*

$$c_\delta \alpha \leq d_\delta(\xi) \leq \frac{1}{4}\alpha^2, \tag{19}$$

we have that $\xi \in \rho(H^{B,\mathbb{M}})$, and

$$\|(H^{B,\mathbb{M}} - \xi)^{-1} - (H_\delta^{B,\mathbb{M}} - \xi)^{-1}\| \leq \frac{C_\delta\, \alpha}{d_\delta(\xi)^2}. \tag{20}$$

See [BD] for weaker versions of this theorem. We also mention [BaSoY], which established[2] the convergence of the ground state energy of fermionic H^B (see below) to that of bosonic (scalar) h_δ^B on $L^2(\mathbb{R}^N)$, using variational arguments: these authors did not fix $\mathbb{M}$, but they only proved convergence of the ground state energy, while we can conclude much more from the norm resolvent convergence to the effective hamiltonians; see §9 for a list of applications of the results of the present paper. Earlier, [LSY] had shown that the ground state of the Hartree mean-field model associated to (16) approximates the quantum mechanical ground state energy in the so-called hyper-strong limit $Z, B/Z^3 \to \infty$, assuming N/Z uniformly bounded. The idea that a model such as the δ-model could be relevant in the context of strong magnetic fields is not new in the physics literature, see e.g. [Spr].

[2][BaSoY], following [LSY], first did a re-scaling of H^B's ground state energy which allowed them to compare with h_δ^B for a *fixed* B (e.g. fixing $2\alpha(B) = 1$). Since this homogeneity property is not valid anymore for our other two effective hamiltonians, we prefer not to do this here (contrary to our earlier papers [BD]), in order to have a coherent presentation.

We next turn to the effects of particle symmetry. Electrons in physical atoms are fermions, and we now consider the analogues of Theorems 1.1, 1.3 and 1.5 for H^B restricted to the fermionic subspace $\mathcal{H}_{\mathrm{f}} = P^{AS}(\mathcal{H})$, where P^{AS} is the orthogonal projection onto the subspace of anti-symmetric symmetric wave-functions defined by:

$$P^{\mathrm{AS}}\psi(r_1, s_1; \cdots ; r_N, s_N) := \frac{1}{N!}\sum_{\sigma\in S_N} (-1)^\sigma \psi\left(r_{\sigma(1)}, s_{\sigma(1)}; \cdots ; r_{\sigma(N)}, s_{\sigma(N)}\right),$$

with $(-1)^\sigma := (-1)^{\mathrm{sgn}\,(\sigma)}$. The projection P^{AS} commutes with H^B, $\mathbb{L}_z$, $\mathbb{S}_z$ and with the N-particle Landau Hamiltonian H_0^B defined in (24) below, and therefore also with $\Pi_{\mathrm{eff}}^{B,\mathbb{M}}$ (see section 2). Recalling that we have fixed our spins to $\mathbb{S}_z = -NB/2$, P^{AS} for us will only act in the 'spatial' variables $(r_1, \cdots , r_N)$. Let $H_{\mathrm{f}}^{B,\mathbb{M}} := P^{\mathrm{AS}} H^{B,\mathbb{M}} P^{\mathrm{AS}} = H^{B,\mathbb{M}} P^{\mathrm{AS}}$, the fermionic Pauli operator with z-angular momentum $\mathbb{M}$. Similarly, introduce 'fermionized' versions of the other operators: $\Pi_{\mathrm{eff,f}}^{B,\mathbb{M}} := P^{\mathrm{AS}}\Pi_{\mathrm{eff}}^{B,\mathbb{M}}$, $\Pi_{\perp,\mathrm{f}}^{B,\mathbb{M}} := P^{\mathrm{AS}}\Pi_{\perp}^{B,\mathbb{M}}$, $H_{\mathrm{eff,f}}^{B,\mathbb{M}} := P^{AS} H_{\mathrm{eff}}^{B,\mathbb{M}} = P^{AS} h_{\mathrm{eff}}^{B,\mathbb{M}} \oplus H_{\perp,\mathrm{f}}^{B,\mathbb{M}} =: h_{\mathrm{eff,f}}^{B,\mathbb{M}} \oplus H_{\perp,\mathrm{f}}^{B,\mathbb{M}}$, $h_{C,\mathrm{f}}^{B,\mathbb{M}} = P^{AS} h_C^{B,\mathbb{M}}$, $h_{\delta,\mathrm{f}}^{B} := P^{AS} h_\delta^B$, $H_{C,\mathrm{f}}^{B,\mathbb{M}} = P^{AS} H_C^{B,\mathbb{M}}$ and finally $H_{\delta,\mathrm{f}}^{B,\mathbb{M}} := P^{AS} H_\delta^{B,\mathbb{M}}$. A careful examination of the proofs of Theorems 1.1, 1.3 and 1.5 will show:

Theorem 1.6. *Theorems 1.1, 1.3 and 1.5 also hold true for the fermionized operators. For example, if ξ satisfies the conditions of theorem 1.1 with $d_{\mathrm{eff}}(\xi)$ replaced by $d_{\mathrm{eff,f}}(\xi) := \mathrm{dist}\left(\xi, \sigma(h_{\mathrm{eff,f}}^{B,\mathbb{M}})\right)$, then*

$$\| \left(H_{\mathrm{f}}^{B,\mathbb{M}} - \xi\right)^{-1} - \left(H_{\mathrm{eff,f}}^{B,\mathbb{M}} - \xi\right)^{-1} \| \leq C_{\mathrm{eff}} \frac{\alpha(B)^2}{d_{\mathrm{eff,f}}(\xi)^2\sqrt{B}}, \tag{21}$$

with the same constants as before. Similarly for $H_{C,\mathrm{f}}^{B,\mathbb{M}}$, $H_{\delta,\mathrm{f}}^{B,\mathbb{M}}$.

Remark 1.7. Theorem 1.6 is not simply obtained by "sandwiching" Theorems 1.1, 1.3 and 1.5 between P^{AS}, since the statements thus obtained would not involve the distances to the spectra of the fermionized operators. Also, we established the fermionic versions with the same constants as for the boltzonic ones, but it is conceivable that one could have smaller constants in the fermionic case.

The operators $h_{\mathrm{eff,f}}^{B,\mathbb{M}}$ and $h_{C,\mathrm{f}}^{B,\mathbb{M}}$ are easily described:

$$h_{\mathrm{eff,f}}^{B,\mathbb{M}} = -\frac{1}{2}\Delta - Z\sum_{j=1}^{N} V_{\mathrm{av:1}}^{B,\mathbb{M}}(z_j) + \sum_{j<k} V_{\mathrm{av:2}}^{B,\mathbb{M}}(z_j - z_k), \tag{22}$$

where, using the notation of (10),

$$V_{\mathrm{av:1}}^{B,\mathbb{M}}(z) := \frac{1}{N}\sum_j V_j^{B,\mathbb{M}}(z),$$

and

$$V_{\mathrm{av:2}}^{B,\mathbb{M}}(z) := \left(\begin{array}{c} N \\ 2 \end{array}\right)^{-1} \sum_{j<k} V_{jk}^{B,\mathbb{M}}(z),$$

the average one-, respectively two-particle potentials. Similarly, $h_{C,\mathrm{f}}^{B,\mathbb{M}}$ equals

$$\begin{aligned} h_{C,\mathrm{f}}^{B,\mathbb{M}} \quad = \quad & -\frac{1}{2}\Delta - Z\sum_{j=1}^{N}\left(q^B(z_j) + C_{\mathrm{av:1}}^{n,B,\mathbb{M}}\delta(z_j) \right) \\ & + \sum_{j<k}\left(q^B(z_j - z_k) + C_{\mathrm{av:2}}^{e,B,\mathbb{M}}\delta(z_j - z_k) \right), \end{aligned} \tag{23}$$

with

$$C_{\mathrm{av}:1}^{n,B,\mathbb{M}} := \frac{1}{N}\sum_{j=1}^{N} C_j^{n,B,\mathbb{M}},$$

and

$$C_{\mathrm{av}:2}^{e,B,\mathbb{M}} := \left(\begin{array}{c} N \\ 2 \end{array}\right)^{-1} \sum_{j<k} C_{jk}^{e,B,\mathbb{M}},$$

while $h_{\delta,\mathrm{f}}^{B,\mathbb{M}}$ is given by the same expression as h_δ^B, except of course that its domain changes.

To complete the picture, we finish with an explicit description of $\mathrm{Ran}(\Pi_{\mathrm{eff,f}}^{B,\mathbb{M}})$, which is equal to $P^{\mathrm{AS}}\left(\mathrm{Ran}\ \Pi_{\mathrm{eff}}^{B,\mathbb{M}}\right) = P^{AS}\left(L^2(\mathbb{R}^N)\otimes F_{\mathbb{M}}^B\right)$. Fix $\mathbb{M}\geq 0$, and let

$$\Sigma(\mathbb{M}) = \{m\in\mathbb{N}^N : |m| = \mathbb{M}\},$$

the set of partitions of $\mathbb{M}$, where, as usual, $|m| = m_1+\cdots+m_N$ if $m = (m_1,\cdots,m_N)$. The vector space $F_{\mathbb{M}}^B$ is spanned by the lowest Landau (generalized) eigenstates indexed by $m\in\Sigma(\mathbb{M})$, cf. section 2 below. The symmetric group S_N acts on $\Sigma(\mathbb{M})$ in the natural way, by permuting the indices of an element $m = (m_1,\cdots,m_N)$ of $\Sigma(\mathbb{M})$. Under this action, $\Sigma(\mathbb{M})$ will decompose as a finite union of disjoint orbits:

$$\Sigma(\mathbb{M}) = \bigcup_{\overline{m}\in\mathcal{M}} S_N\cdot\overline{m},$$

$\mathcal{M}$ being a set of representatives of the orbits. If $G_{\overline{m}}$ denotes the stabilizer of $\overline{m}\in\mathcal{M}$, then we write $\mathcal{M}$ as a disjoint union $\mathcal{M} = \mathcal{M}_1\cup\mathcal{M}_2$, with $\mathcal{M}_1$ the subset of those $\overline{m}\in\mathcal{M}$ such that $G_{\overline{m}} = \{e\}$, and $\mathcal{M}_2 = \mathcal{M}\setminus\mathcal{M}_1$, its complement. In other words, $\overline{m}\in\mathcal{M}_1$ iff no two components of $\overline{m}$ are the same, and $\overline{m}\in\mathcal{M}_2$ iff at least two of its components are identical. Let $L^2_{\mathrm{AS}}(\mathbb{R}^N)$ be the space of anti-symmetrical wave functions in $L^2(\mathbb{R}^N)$. We then will prove, in section 8, that:

Theorem 1.8. *There is a natural unitary isomorphism*

$$U_{\mathbb{M}}^B : \mathrm{Ran}(\Pi_{\mathrm{eff,f}}^{B,\mathbb{M}}) \to \sum_{\overline{m}\in\mathcal{M}_1}^{\oplus} L^2(\mathbb{R}^N)\oplus\sum_{\overline{m}\in\mathcal{M}_2}^{\oplus} L^2_{AS}(\mathbb{R}^N),$$

and

$$U_{\mathbb{M}}^B\ h_{\delta,\mathrm{f}}^{B,\mathbb{M}}\ U_{\mathbb{M}}^{B*} = \sum_{\overline{m}\in\mathcal{M}_1}^{\oplus} h_\delta^B|_{L^2(\mathbb{R}^N)}\oplus\sum_{\overline{m}\in\mathcal{M}_2}^{\oplus} h_\delta^B|_{L^2_{\mathrm{AS}}(\mathbb{R}^N)}.$$

Remark 1.9. The operators $U_{\mathbb{M}}^B\ h_{C,\mathrm{f}}^{B,\mathbb{M}}\ U_{\mathbb{M}}^{B*}$ and $U_{\mathbb{M}}^B\ h_{\mathrm{eff,f}}^{B,\mathbb{M}}\ U_{\mathbb{M}}^{B*}$ can mix different components of Ran $(U_{\mathbb{M}}^B)$, as we will see at the end of section 8, and will therefore have a more complicated structure.

The paper is organized as follows. Section 2 contains the precise definition of our effective projector $\Pi_{\mathrm{eff}}^{B,\mathbb{M}}$. In section 3 we establish, with the help of the Feschbach decomposition, a first approximation theorem, comparing $H^{B,\mathbb{M}}$'s resolvent at ξ with that of $H_{\mathrm{eff}}^{B,\mathbb{M}}+\mathcal{W}^{B,\mathbb{M}}(\xi)$, where the last term is an auxiliary 'potential' which itself depends on the spectral parameter ξ. Section 4 analyzes the large-B behavior of the potential of $H_{\mathrm{eff}}^{B,\mathbb{M}}$, as well as that of $\mathcal{W}^{B,\mathbb{M}}$. Sections 5, 6 and 7 are devoted to the proofs of, respectively, theorems 1.1, 1.3 and 1.5. In section 8 we prove theorems 1.6 and 1.8. Section 9, finally, concludes with some applications to the spectral theory of $H^{B,\mathbb{M}}$, and some general observations.

Convention on constants. In the course of this work, we have had to introduce a large number of constants. To keep track of them, we will use the convention that

whenever the subscript of a constant is a number, the number refers to the formula where the constant in question was first introduced. That is, $C_{(x)}$:= constant defined in formula (x).

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2. Non-interacting electrons and the Lowest Landau Band

We begin by reviewing the spectral decomposition of

$$H_0^B := H_0^B(N) := \sum_{j=1}^N \frac{1}{2}\left((\frac{1}{i}\nabla_{r_j} - \frac{1}{2}\mathbb{B}\wedge r_j)^2 - NB \right), \tag{24}$$

the "free" Hamiltonian of N independent electrons interacting only with the field $\mathbb{B}$. Recall that we have fixed all electron spins in their $s_{z_j} = -1$-state. The operator H_0^B is just a direct sum of N one-particle operators $\frac{1}{2}\left((i^{-1}\nabla_r - \frac{1}{2}\mathbb{B}\wedge r)^2 - B\right)$, whose spectral decomposition is explicitly known:

$$\bigoplus_{m\in\mathbb{Z}, n\in\mathbb{N}} \left(\frac{1}{2}p_z^2 + \frac{B}{2}(2n + |m| - m) \right) \Pi_{m,n}^B .$$

Here, p_z is the momentum in the field direction, and $\Pi_{m,n}^B$ is the projection, in the x, y-variables, onto the normalized eigenfunctions $\chi_{m,n}^B = \chi_{m,n}^B(x,y) \in L^2(\mathbb{R}^2)$ of the operator

$$-\frac{1}{2}\Delta_{x,y} + \frac{B^2}{8}(x^2+y^2) - \frac{B}{2},$$

restricted to the m-eigenspace of $L_z = xp_y - yp_x$, the angular momentum in the field-direction. These eigenfunctions are explicitly known in terms of Laguerre functions, see e.g. [FW], but for our purposes we will only need those with $n = 0$, $m \geq 0$. These have a particularly simple expression: if (ρ, φ) are polar coordinates in the x, y-plane, then

$$\chi_m^B := \chi_{m,0}^B : (x,y) \to \left(\frac{B^{m+1}}{2\pi 2^m m!} \right)^{1/2} \rho^m e^{+im\varphi} e^{-B\rho^2/4}. \tag{25}$$

The spectral decomposition of H_0^B is simply the sum of the one-particle decompositions, and the projections onto its eigenstates will be indexed by N-tuples $m = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, $n = (n_1, \cdots n_N) \in \mathbb{N}^N$. If we let

$$\Pi_{m,n}^B := \Pi_{m_1,n_1}^B \otimes \cdots \otimes \Pi_{m_N,n_N},$$

then

$$H_0^B = \bigoplus_{m,n} \left(\sum_{j=1}^N \frac{1}{2}p_{z_j}^2 + \frac{B}{2}(2n_j + |m_j| - m_j) \right) \Pi_{m,n}^B . \tag{26}$$

The Lowest Landau Band of H_0^B is defined as

$$\mathcal{L}_0^B = \bigoplus_{m\in\mathbb{Z}_+^N} \mathrm{Ran}\ \Pi_{m,\mathbf{0}}^B , \tag{27}$$

where $\mathbf{0} := (0, \cdots, 0)$, and if we put

$$X_m^B(x,y) := \prod_{j=1}^N \chi_{m_j}^B(x_j, y_j), \qquad m_1, \cdots, m_N \geq 0, \tag{28}$$

then $\mathcal{L}_0$ will be spanned by the tensor products $X_m^B \otimes u$, with $u = u(z) \in L^2(\mathbb{R}^N)$. We will call the X_m^B the lowest Landau band states (these are not eigenvectors of H_0^B, but $X_m^B \otimes 1$ would be generalized eigenvectors with eigenvalue 0). The operator H_0^B restricted to $\mathcal{L}_0^B$ simply is the free Laplacian in the field direction,

$$\frac{1}{2}\sum_j p_{z_j}^2 = -\frac{1}{2}\Delta_z,$$

where $z = (z_1, \cdots, z_N)$.

We next reduce the Hamiltonians H^B and H_0^B to their angular momentum sectors with respect to the field direction. The total orbital angular momentum in the direction of $\mathbb{B} = (0, 0, B)$,

$$\mathbb{L}_z = \sum_j (x_j p_{y_j} - y_j p_{x_j}), \quad (p_x, p_y, p_z) = \frac{1}{i}\nabla_r,$$

commutes with H^B and H_0^B, and is therefore a constant of motion for both Hamiltonians. If $P^{\mathbb{M}}$ is the orthogonal projection onto the $\mathbb{M}$-th eigenspace of $\mathbb{L}_z$, then we let

$$H^{B,\mathbb{M}} := H^B P^{\mathbb{M}}, \qquad H_0^{B,\mathbb{M}} := H_0^B P^{\mathbb{M}} \tag{29}$$

acting on $\mathcal{H} := L^2(\mathbb{R}^{3N}) \otimes \mathbb{C}^{2N}$. Since we are primarily interested in the spectral behavior of H^B near the bottom of its spectrum, we will restrict $\mathbb{M}$ to $\mathbb{Z}_+$, for $\mathcal{L}_0^B \cap \text{Ran } P^{\mathbb{M}} \neq \{0\} \Leftrightarrow \mathbb{M} \geq 0$. Indeed, notice that since $H^{B,-\mathbb{M}}$ is unitarily equivalent to $H^{B,\mathbb{M}} + \mathbb{M}B$, one has $\inf \sigma(H^B|_{\mathbb{M}\geq 0}) < \inf \sigma(H^B|_{\mathbb{M}<0})$ as soon as $B > 0$.

We next let

$$\Sigma(\mathbb{M}) = \{m = (m_1, \cdots m_N) \in \mathbb{Z}^N : m_j \geq 0, \ m_1 + \cdots + m_N = \mathbb{M}\}, \tag{30}$$

the set of partitions of $\mathbb{M}$, and define the *effective projection* $\Pi_{\text{eff}}^{B,\mathbb{M}}$ by

$$\Pi_{\text{eff}}^{B,\mathbb{M}} := \sum_{m \in \Sigma(\mathbb{M})} \Pi_{m,\mathbf{0}}^B. \tag{31}$$

This is simply the orthogonal projection onto $\mathcal{L}_0 \cap (\mathbb{L}_z = \mathbb{M}\}$. We also let $\Pi_\perp^{B,\mathbb{M}}$ be the orthogonal projection onto the orthogonal complement of $\text{Ran}(\Pi_{\text{eff}}^{B,\mathbb{M}})$ in $\text{Ran}(P^{\mathbb{M}})$. Observe that

$$\Pi_\perp^{B,\mathbb{M}} = \bigoplus_{\substack{m_1 + \ldots + m_N = \mathbb{M}, \\ \sum_j 2n_j + |m_j| - m_j \geq 2}} \Pi_{m,n}.$$

If we let $F_{\mathbb{M}}^B$ be the finite dimensional vector space spanned by the lowest Landau states with total angular momentum $\mathbb{M}$,

$$F_{\mathbb{M}}^B := \text{Span}\{X_m^B : m \in \Sigma(\mathbb{M})\}, \tag{32}$$

then we can identify the range of $\Pi_{\text{eff}}^{B,\mathbb{M}}$ with the space $L^2(\mathbb{R}^N, F_{\mathbb{M}}^B)$ of $F_{\mathbb{M}}^B$-valued L^2-functions, as we will do without further comment.

To lighten the notations, we will often suppress one or both upper-indices B or $\mathbb{M}$, unless where this would cause confusion. This will always be clearly indicated, usually at the beginning of a section.

3. Estimates for Feschbach decompositions

We fix a non-negative integer $\mathbb{M} \geq 0$. In this section we will drop all upper-indices B, $\mathbb{M}$, and simply write H for $H^{B,\mathbb{M}}$, H_0 for $H_0^{B,\mathbb{M}}$ and $\Pi_{\rm eff}$ respectively $\Pi_\perp$ for $\Pi_{\rm eff}^{B,\mathbb{M}}$ and $\Pi_\perp^{B,\mathbb{M}}$. We write our atomic Hamiltonian $H^{B,\mathbb{M}}$ as

$$H = H_0 + \mathcal{V},$$

where

$$\mathcal{V} := -\sum_j \frac{Z}{|r_j|} + \sum_{j<k} \frac{1}{|r_j - r_k|}, \tag{33}$$

the electrostatic potential, and introduce the operators

$$\mathcal{V}_{\rm eff} := \Pi_{\rm eff}\mathcal{V}\Pi_{\rm eff},\ \mathcal{V}_\perp := \Pi_\perp\mathcal{V}\Pi_\perp,\ \mathcal{V}_{\perp,{\rm eff}} := \Pi_\perp\mathcal{V}\Pi_{\rm eff},$$

and its adjoint, $\mathcal{V}_{{\rm eff},\perp} = \Pi_{\rm eff}\mathcal{V}\Pi_\perp$. These are to be considered as operators on $\mathrm{Ran}\ \Pi_{\rm eff}$, $\mathrm{Ran}\ \Pi_\perp$, and between these two Hilbert spaces, respectively. We furthermore put

$$h_{\rm eff} := \Pi_{\rm eff} H \Pi_{\rm eff}\ , H_\perp := \Pi_\perp H \Pi_\perp,\ H_{\perp,{\rm eff}} := \Pi_\perp H \Pi_{\rm eff} = H^*_{{\rm eff},\perp},$$

and for $\xi \in \mathbb{C}$ introduce the resolvents (wherever defined)

$$R := R(\xi) := (H_\perp - \xi)^{-1}\ ,\ R^{\mathcal{W}}_{\rm eff} := R^{\mathcal{W}}_{\rm eff}(\xi) := (h_{\rm eff} + \mathcal{W}(\xi) - \xi)^{-1}\ ,$$

where

$$\mathcal{W} := \mathcal{W}(\xi) = -\mathcal{V}_{{\rm eff},\perp} R(\xi) \mathcal{V}_{\perp,{\rm eff}}.$$

Strictly speaking $R^{\mathcal{W}}_{\rm eff}$ is not a resolvent since the potential $\mathcal{W}(\xi)$ depends on the spectral parameter ξ. The operators R and $R^{\mathcal{W}}_{\rm eff}$ act on, respectively, the ranges of $\Pi_\perp$ and of $\Pi_{\rm eff}$. Finally, let

$$T := H_0 P^{\mathbb{M}},\ T_{\rm eff} := \Pi_{\rm eff} T \Pi_{\rm eff},\ T_\perp := \Pi_\perp T \Pi_\perp;$$

T commutes with $\Pi_{\rm eff}$ and $\Pi_\perp$, and $\Pi_\perp T \Pi_{\rm eff} = 0$. Note that

$$T_\perp \geq B\ \Pi_\perp,$$

on the range of $P^{\mathbb{M}}$.

Using matrix notation associated to the decomposition $P^{\mathbb{M}}H = \mathrm{Ran}\ \Pi_{\rm eff} \oplus \mathrm{Ran}\ \Pi_\perp$, we decompose H as

$$H \quad = \quad \begin{pmatrix} h_{\rm eff} & H_{{\rm eff},\perp} \\ H_{\perp,{\rm eff}} & H_\perp \end{pmatrix} = \begin{pmatrix} T_{\rm eff} + \mathcal{V}_{\rm eff} & \mathcal{V}_{{\rm eff},\perp} \\ \mathcal{V}_{\perp,{\rm eff}} & T^B_\perp + \mathcal{V}_\perp) \end{pmatrix}. \tag{34}$$

By the classical Feschbach formula, we then have

$$(H-\xi)^{-1} = \begin{pmatrix} R^{\mathcal{W}}_{\rm eff} & -R^{\mathcal{W}}_{\rm eff}\mathcal{V}_{{\rm eff},\perp}R \\ -R\mathcal{V}_{\perp,{\rm eff}}R^{\mathcal{W}}_{\rm eff} & R + R\mathcal{V}_{\perp,{\rm eff}}R^{\mathcal{W}}_{\rm eff}\mathcal{V}_{{\rm eff},\perp}R \end{pmatrix}, \tag{35}$$

for those $\xi \in \mathbb{C}$ for which the right hand side makes sense. The following theorem is the main result of this section: recall that $\rho(A)$ denotes the resolvent set of an operator A, and $\sigma(A)$ its spectrum.

Theorem 3.1. *Let*

$$B_{(36)} := 16Z^2N(\mathbb{M} + N + 2), \tag{36}$$

and

$$C_{(37)} := c_0 + \frac{c_0^2}{\sqrt{B_{(36)}}}, \qquad c_0^2 = (32Z^2N + 8N(N-1)^2)(\mathbb{M} + N + 2). \tag{37}$$

If $\xi \le 0$ and if the field strength $B \ge B_{(36)}$, then $\xi \in \rho(H_\perp)$. If, moreover, $\xi \notin \sigma(H_{\mathrm{eff}} + \mathcal{W})$, then $\xi \in \rho(H)$, and

$$\| \, (H-\xi)^{-1} - (h_{\mathrm{eff}} + \mathcal{W} - \xi)^{-1} \oplus R(\xi) \| \le \frac{C_{(37)}}{d_{\mathrm{eff}}^{\mathcal{W}}(\xi)\sqrt{B}}. \tag{38}$$

where $d_{\mathrm{eff}}^{\mathcal{W}}(\xi)$ is the distance of ξ to $\sigma(h_{\mathrm{eff}} + \mathcal{W})$.

Proof of Theorem 3.1. The proof consists of estimating the relevant matrix elements in the Feschbach formula. This will be done in several steps. Let

$$R_0 = (T_\perp - \xi)^{-1},$$

the resolvent of $T_\perp$ on Ran $\Pi_\perp$.

Bound on R_0. Since $T_\perp \ge B$ on Ran $\Pi_\perp$ and since $\xi \le 0$, it immediately follows that $\|R_0\| \le B^{-1}$. Write $\mathcal{V}$ as

$$\mathcal{V} = \mathcal{V}_n + \mathcal{V}_e, \quad \text{where} \quad \mathcal{V}_n := -\sum_j \frac{Z}{|r_j|}, \quad \text{and} \quad \mathcal{V}_e := \sum_{j<k} \frac{1}{|r_j - r_k|}, \tag{39}$$

the sum of electron-nucleus and the electron-electron interactions.

A remark on notation: we will often leave the projection $\Pi_\perp$ understood when multiplying operators on the left and/or right by R_0 or R, and for example simply write $R_0^{1/2}\mathcal{V}_n R_0^{1/2}$ instead of the more explicit $R_0^{1/2}\Pi_\perp \mathcal{V}_n \Pi_\perp R_0^{1/2}$.

Bound on $R_0^{1/2}\mathcal{V}_n R_0^{1/2}$. First, since $R_0 \le B^{-1}$ on the range of $\Pi_\perp$,

$$0 \le (\sqrt{R_0}\mathcal{V}_n\sqrt{R_0})^2 \le B^{-1}\sqrt{R_0}\mathcal{V}_n^2\sqrt{R_0}.$$

By Cauchy-Schwarz,

$$\sqrt{R_0}\mathcal{V}_n^2\sqrt{R_0} \le Z^2 N \sqrt{R_0}\sum_j |r_j|^{-2}\sqrt{R_0}.$$

Next, using Hardy's inequality on $\mathbb{R}^3$: $|r_j|^{-2} \le -4\Delta_j$, and the fact that

$$H_0^B = \sum_j -\frac{1}{2}\Delta_j + \frac{1}{8}|\mathbb{B} \wedge r_j|^2 - \frac{B}{2}(\mathbb{L}_z + N),$$

we find that

$$\begin{aligned}
\sqrt{R_0}\left(\sum_j |r_j|^{-2}\right)\sqrt{R_0} &\le 8\sqrt{R_0}\sum_j \left(-\frac{\Delta_j}{2} + \frac{|\mathbb{B}\wedge r_j|^2}{8}\right)\sqrt{R_0} \\
&= 8\sqrt{R_0}\left(T_\perp + \frac{B}{2}(\mathbb{M}+N)\right)\sqrt{R_0} \\
&= 8\Pi_\perp + 8\xi R_0 + 4(\mathbb{M}+N)BR_0 \\
&\le (8 + 4(\mathbb{M}+N))\,\Pi_\perp,
\end{aligned}$$

since $\xi \le 0$. It follows from these estimates that

$$\|R_0^{1/2}\mathcal{V}_n R_0^{1/2}\| \le 2ZB^{-1/2}\sqrt{N(\mathbb{M}+N+2)}.$$

We note, as a consequence, that if $b_0 = 4Z^2N(\mathbb{M}+N+2)$, then $\|R_0^{1/2}\mathcal{V}_n R_0^{1/2}\| \le (b_0 B^{-1})^{1/2} < 1$ if $B > b_0$. For later reference we also note the:

Bound on $R_0^{1/2}\mathcal{V}_n^2 R_0^{1/2}$: the estimates above immediately imply that this positive operator is bounded from above by $4Z^2N(\mathbb{M}+N+2)$.

Existence of and bound on R. Since the electron-electron repulsion $\mathcal{V}_e \geq 0$, it follows that $R \leq R_{\rm NI}$, where $R_{\rm NI} = (T_\perp + \mathcal{V}_{n,\perp} - \xi)^{-1}$, the resolvent of an atom with non-interacting electrons. Using the symmetrized resolvent formula,

$$R_{\rm NI} = \sqrt{R_0}\left(1 + \sqrt{R_0}\mathcal{V}_n\sqrt{R_0}\right)^{-1}\sqrt{R_0}. \tag{40}$$

we see that $R_{\rm NI}$ exists and is positive if $B > b_0$ and $\xi \leq 0$. Hence $T_\perp + \mathcal{V}_{n,\perp} - \xi \geq 0$ and therefore also $H_\perp - \xi$ and R. Moreover, if $B > 4b_0 = 16Z^2N(\mathbb{M} + N + 2) = B_{(36)}$, then every $\xi \leq 0$ belongs to $\rho(H_\perp)$ and

$$0 \leq R \quad \leq \quad \frac{\|R_0\|}{1 - \|R_0^{1/2}\mathcal{V}_n R_0^{1/2}\|} \leq B^{-1}\left(\frac{1}{1 - (b_0 B^{-1})^{1/2}}\right) \leq 2B^{-1}.$$

Bound on $R_0^{1/2}\mathcal{V}_e^2 R_0^{1/2}$: The following elementary operator inequality is very useful to estimate the electron-electron interactions.

Lemma 3.2.

$$\frac{1}{|r_j - r_k|^2} \leq 2\left(-\Delta_j - \Delta_k\right). \tag{41}$$

Proof. The unitary transformation induced by the following orthogonal transformation of $\mathbb{R}^3 \times \mathbb{R}^3$,

$$s := \frac{r_1 - r_2}{\sqrt{2}}, \quad t := \frac{r_1 + r_2}{\sqrt{2}}. \tag{42}$$

commutes with the Laplacian, and transforms $|r_j - r_k|^{-2}$ into $2^{-1}|s|^{-2}$. By Hardy's inequality,

$$\frac{1}{2|s|^2} \leq -2\Delta_s \leq -2(\Delta_s + \Delta_t),$$

and transforming back to the (r_j, r_k)-coordinates yields (41). QED

We can then estimate:

$$\begin{aligned}
\mathcal{V}_e^2 &= \left(\sum_{i<j}\frac{1}{|r_i - r_j|}\right)^2 \leq \frac{N(N-1)}{2}\sum_{i<j}\frac{1}{|r_i - r_j|^2} \\
&\leq N(N-1)\sum_{i<j}(-\Delta_i - \Delta_j) = N(N-1)^2\sum_i(-\Delta_i) \\
&\leq 2N(N-1)^2\sum_i\left(-\frac{\Delta_i}{2} + \frac{|\mathbb{B}\wedge r_j|^2}{8}\right),
\end{aligned}$$

and therefore, by similar arguments as before,

$$\begin{aligned}
\sqrt{R_0}\mathcal{V}_e^2\sqrt{R_0} &\leq 2N(N-1)^2\sqrt{R_0}(T_\perp + \frac{B}{2}(\mathbb{M} + N))\sqrt{R_0} \\
&\leq N(N-1)^2(\mathbb{M} + N + 2),
\end{aligned}$$

on Ran $(\Pi_\perp)$.

Bound on $\mathcal{V}R_0^{1/2}$: By the general identity $\|AA^*\| = \|A\|^2$, we have:

$$\begin{aligned}
\|\mathcal{V}R_0^{1/2}\|^2 &= \|R_0^{1/2}\mathcal{V}^2R_0^{1/2}\| \leq 2(\|R_0^{1/2}\mathcal{V}_n^2R_0^{1/2}\| + \|R_0^{1/2}\mathcal{V}_e^2R_0^{1/2}\|) \\
&\leq (8Z^2N + 2N(N-1)^2)(\mathbb{M} + N + 2).
\end{aligned}$$

Bound on $\mathcal{V}_{\mathrm{eff},\perp}R^{1/2}$: (Remember that we have shown that $R \geq 0$, so its square root is well-defined.) We first estimate $\|\mathcal{V}R\mathcal{V}\|$, as follows. Recalling the non-interacting resolvent R_{NI} introduced above, we have that:

$$0 \leq \mathcal{V}R\mathcal{V} \leq \mathcal{V}R_{\mathrm{NI}}\mathcal{V} = \mathcal{V}R_0^{1/2}\left(1 + R_0^{1/2}\mathcal{V}_n R_0^{1/2}\right)^{-1} R_0^{1/2}\mathcal{V}.$$

Hence its norm can be estimated by:

$$\|\mathcal{V}R\mathcal{V}\| \leq \frac{\|\mathcal{V}R_0^{1/2}\|^2}{1 - \|R_0^{1/2}\mathcal{V}_n R_0^{1/2}\|},$$

from which we obtain an estimate for $\|\mathcal{V}R^{1/2}\|$ by taking square roots. Therefore, if $B \geq B_{(36)} = 4b_0$ as above,

$$\|\mathcal{V}_{\mathrm{eff},\perp}R^{1/2}\| \leq \|\mathcal{V}R^{1/2}\| \leq \sqrt{(16Z^2N + 4N(N-1)^2)\,(\mathbb{M} + N + 2)}.$$

We now come to the proof of (38). By Feschbach's formula, we have

$$\begin{aligned} \| \, (H-\xi)^{-1} - R_{\mathrm{eff}}^{\mathcal{W}} \oplus R(\xi)\| &= \left\| \begin{pmatrix} 0 & -R_{\mathrm{eff}}^{\mathcal{W}}\mathcal{V}_{\mathrm{eff},\perp}R \\ -R\mathcal{V}_{\perp,\mathrm{eff}}R_{\mathrm{eff}}^{\mathcal{W}} & R\mathcal{V}_{\perp,\mathrm{eff}}R_{\mathrm{eff}}^{\mathcal{W}}\mathcal{V}_{\mathrm{eff},\perp}R \end{pmatrix} \right\| \\ &\leq \|R_{\mathrm{eff}}^{\mathcal{W}}\mathcal{V}_{\mathrm{eff},\perp}R\| + \|R\mathcal{V}_{\perp,\mathrm{eff}}R_{\mathrm{eff}}^{\mathcal{W}}\mathcal{V}_{\mathrm{eff},\perp}R\| \\ &\leq \|R_{\mathrm{eff}}^{\mathcal{W}}\| \, \|\mathcal{V}_{\mathrm{eff},\perp}R^{1/2}\| \, \|R^{1/2}\| + \|R_{\mathrm{eff}}^{\mathcal{W}}\| \, \|\mathcal{V}_{\mathrm{eff},\perp}R^{1/2}\|^2 \, \|R^{1/2}\|^2, \end{aligned} \tag{43}$$

where we have used the following elementary estimate for the norm of matrices of operators:

$$\| \begin{pmatrix} 0 & A \\ A^\star & B \end{pmatrix} \| \leq \|A\| + \|B\|.$$

Hence if $B > B_{(36)}$ and if $\xi \notin \sigma(R_{\mathrm{eff}}^{\mathcal{W}})$, we obtain

$$\begin{aligned} \| \, (H-\xi)^{-1} - R_{\mathrm{eff}}^{\mathcal{W}} \oplus R(\xi)\| &\leq \frac{1}{d_{\mathrm{eff}}^{\mathcal{W}}(\xi)}\left(\frac{\sqrt{2}\|\mathcal{V}R^{1/2}\|}{\sqrt{B}} + \frac{2\|\mathcal{V}R^{1/2}\|^2}{B}\right) \\ &\leq \frac{1}{d_{\mathrm{eff}}^{\mathcal{W}}(\xi)B^{1/2}}\left(c_0 + \frac{c_0^2}{\sqrt{B_{(36)}}}\right), \end{aligned}$$

with $c_0^2 = (32Z^2N + 8N(N-1)^2)(\mathbb{M} + N + 2)$. QED

Corollary 3.3. (of the proof) *Theorem 3.1 also holds, if we replace H and $h_{\mathrm{eff}} + \mathcal{W}$ by their fermionized versions H_{f}, $h_{\mathrm{eff,f}} + \mathcal{W}_{\mathrm{f}}$, and $d_{\mathrm{eff}}^{\mathcal{W}}(\xi)$ by the distance of ξ to the spectrum of $h_{\mathrm{eff,f}} + \mathcal{W}_{\mathrm{f}}$.*

Proof. Simply write down the Feschbach's formula (35) for H_{f} with respect to the decomposition $I = \Pi_{\mathrm{eff,f}} + \Pi_{\perp,\mathrm{f}}$ of $\mathcal{H}_{\mathrm{f}}$, and estimate as in (43), where all operators will now have a sub-index 'f'. Next use that P^{AS} commutes with everything, and trivially estimate $\|A_{\mathrm{f}}\| = \|P^{AS}A\| \leq \|A\|$, for $A = R, R^{1/2}$ and $\mathcal{V}_{\mathrm{eff},\perp}R^{\frac{1}{2}}$, except for $\|R_{\mathrm{eff,f}}^{\mathcal{W}}\|$, which will be estimated by 1 over the distance of ξ to the spectrum of $H_{\mathrm{eff,f}} + \mathcal{W}_{\mathrm{f}}$. QED

The proof shows that in the fermionic case, Theorem 3.1 will at least be true with the same constants as for the boltzonic case. The optimal constants for fermions might be smaller, though.

Remark 3.4. In the proof of theorem3.1 we systematically used Hardy's inequality. Alternatively, one can use, at least when $N = 1$, the bounds on the matrix elements of the Coulomb potential with respect to the Landau levels which were derived in [FW].

4. Effective potentials for large fields

The operator $h_{\text{eff}} + \mathcal{W} = h_{\text{eff}}^{B,\mathbb{M}} + \mathcal{W}^{B,\mathbb{M}}$ of Theorem 3.1 acts on $\text{Ran}(\Pi_{\text{eff}}^{B,\mathbb{M}}) = \Pi_{\text{eff}}^{B,\mathbb{M}}(\mathcal{H})$, a Hilbert space which depends on both B and $\mathbb{M}$, and which is canonically isomorphic to the space of $F_{\mathbb{M}}^{B}$-valued L^2-functions on $\mathbb{R}^N$,

$$\text{Ran}\ (\Pi_{\text{eff}}^{B,\mathbb{M}}) = L^2(\mathbb{R}^N, F_{\mathbb{M}}^{B}). \tag{44}$$

(Recall that $F_{\mathbb{M}}^{B} = \text{Span}\,\{X_m^B : m \in \Sigma(\mathbb{M})\}$.) We will mostly suppress the $\mathbb{M}$-dependence from our notations, $\mathbb{M}$ being fixed in our analysis. The potential term of h_{eff}^{B},

$$\mathcal{V}_{\text{eff}}^{B} = \Pi_{\text{eff}}^{B} \mathcal{V} \Pi_{\text{eff}}^{B}, \tag{45}$$

can be interpreted as an operator valued function of $z = (z_1, \cdots, z_N) \in \mathbb{R}^N$, with values in the space of linear operators on $F_{\mathbb{M}}^{B}$ and acting in the natural way on $L^2(\mathbb{R}_z^N; F_{\mathbb{M}}^{B})$. To get rid of the B-dependence of our Hilbert spaces we do a unitary re-scaling. Let us pose $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$ and similarly for y and z. Define a unitary operator U_{xy}^{B} on $\mathcal{H}$ by:

$$U_{xy}^{B}\psi(x, y, z) = B^{N/2}\psi(\sqrt{B}x, \sqrt{B}y, z). \tag{46}$$

Since $X_m^B(x, y) = B^{N/2} X_m^1(\sqrt{B}x, \sqrt{B}y)$, it follows that

$$U_{xy}^{B*} \Pi_{\text{eff}}^{B} U_{xy}^{B} = \Pi_{\text{eff}}^{1}.$$

Let us write $\mathcal{V}_{\text{eff}}^{B}$ in multi-particle form:

$$\mathcal{V}_{\text{eff}}^{B} = -\sum_j Z V_j^B + \sum_{j<k} V_{jk}^{B}, \tag{47}$$

with V_j^B and V_{jk}^B defined by (10). Then $U_{xy}^{B*} V_j^B U_{xy}^{B} = \Pi_{\text{eff}}^{1} U_{xy}^{B*} |r_j|^{-1} U_{xy}^{B} \Pi_{\text{eff}}^{1} = \sqrt{B} V_j^1(\sqrt{B} z_j)$, with

$$V_j^1(z) = \Pi_{\text{eff}}^{1} \frac{1}{\sqrt{x_j^2 + y_j^2 + z^2}} \Pi_{\text{eff}}^{1}, \tag{48}$$

and likewise for V_{jk}^B: $U_{xy}^{B*} V_{jk}^B U_{xy}^B(z) = \sqrt{B} V_{jk}^1(\sqrt{B}(z_j - z_k))$, with

$$V_{jk}^1(z) = \Pi_{\text{eff}}^{1} \frac{1}{\sqrt{(x_j - x_k)^2 + (y_j - y_k)^2 + z^2}} \Pi_{\text{eff}}^{1}. \tag{49}$$

The operator

$$\widehat{h}_{\text{eff}}^{B} := U_{xy}^{B*} H_{\text{eff}}^{B} U_{xy}^{B}, \tag{50}$$

will now act on the fixed, B-independent, Hilbert space, $L^2(\mathbb{R}^N, F_{\mathbb{M}}^{1})$, and

$$\begin{aligned} \widehat{h}_{\text{eff}}^{B} &= -\frac{1}{2}\Delta_z - \sum_j Z\sqrt{B} V_j^1(\sqrt{B} z_j) + \sum_{j<k} \sqrt{B} V_{jk}^1(\sqrt{B}(z_j - z_k)) \\ &= -\frac{1}{2}\Delta_z + \sqrt{B}\mathcal{V}_{\text{eff}}^{1}(\sqrt{B} z). \end{aligned}$$

The next step will be to examine the asymptotic behavior of $\sqrt{B}\mathcal{V}_{\text{eff}}^{1}(\sqrt{B}z)$ as $B \to \infty$. The main idea is contained in lemma 4.1 below. We introduce the free Laplacian on $\mathbb{R}^N$,

$$h_{00} = -\frac{1}{2}\Delta_z, \tag{51}$$

and its resolvent:

$$R_{00}(-\alpha^2) = (h_{00} + \alpha^2)^{-1}. \tag{52}$$

We will need this resolvent both in dimension N and dimension 1. To distinguish between these two cases we will, in the 1-dimensional case, systematically use β^2 as spectral parameter instead of α^2, reserving the latter for the multidimensional case.

If u is a function or tempered distribution on $\mathbb{R}^N$, with values in some auxiliary Hilbert space F, then $\|R_{00}(-\alpha^2)^{s/2}u\|_{L^2(\mathbb{R}^N;F)}$ is a norm on the s-th Sobolev space $H^s(\mathbb{R}^N;F)$. A linear operator A sends $H^s(\mathbb{R}^N;F)$ continuously into $H^{-s}(\mathbb{R}^N;F)$ iff the L^2-operator norm $\|R_{00}(-\alpha^2)^{s/2}AR_{00}(-\alpha^2)^{s/2}\|$ is finite. The case of interest for us will be $s=1$. We will also need the Fourier transform $\mathcal{F}$, but only in dimension 1, for which we normalize as follows:

$$\mathcal{F}(u)(\zeta) = \int_{\mathbb{R}} u(z)e^{-iz\zeta}dz.$$

There will consequently be a factor of $(2\pi)^{-1}$ in the inversion formula.

Recall that

$$\mathrm{Pf}\left(\frac{1}{|x|}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{sgn}(x)\log|x|\right),$$

with the derivative in distribution sense. Let F be a finite dimensional complex Hilbert space, and $L(F)$ the space of linear operators on F.

Lemma 4.1. *Let $\mathfrak{v}$ be an $L(F)$-valued tempered distribution on $\mathbb{R}$, such that its Fourier transform can be identified with a locally integrable function $\mathcal{F}\mathfrak{v} = \mathcal{F}\mathfrak{v}(\zeta)$. Assume also:*

(i) There exist $C_0, C_1 \in L(F)$ and $a > 1/2$, such that:

$$\mathcal{F}\mathfrak{v}(\zeta) = -C_0\log|\zeta| + C_1 + O(|\zeta|^a), \quad \zeta \to 0. \tag{53}$$

(ii) If

$$\mathfrak{e}(\zeta) := \mathcal{F}\mathfrak{v}(\zeta) + C_0\log|\zeta| - C_1,$$

denotes the error in the approximation (53), then

$$C_{\mathfrak{v}}^2 := \int_{\mathbb{R}} \frac{\|\mathfrak{e}(\zeta)\|^2}{|\zeta|^2}d\zeta < \infty. \tag{54}$$

For each $\lambda > 0$ let

$$\mathfrak{v}_{\infty,\lambda} := C_0\log\lambda\cdot\delta + \frac{1}{2}C_0\cdot\mathrm{Pf}\left(\frac{1}{|x|}\right) + (\gamma C_0 + C_1)\cdot\delta, \tag{55}$$

where δ is Dirac's delta-distribution in 0, and $\gamma = \Gamma'(1)$ is the Euler constant. If $R_{00}(-\beta^2)$ denotes the free resolvent in dimension 1 and $\beta > 0$, then

$$\|R_{00}(-\beta^2)^{1/2}\left(\lambda\mathfrak{v}(\lambda\cdot) - \mathfrak{v}_{\infty,\lambda}\right)R_{00}(-\beta^2)^{1/2}\| \le \frac{2^{1/4}C_{\mathfrak{v}}}{\sqrt{\beta\lambda\pi}}. \tag{56}$$

Remark. Observe that the integral (54) converges in 0, since we assumed that $a > 1/2$ in (53).

Proof. It is known that

$$\mathcal{F}^{-1}(\log|\zeta|) = -\frac{1}{2}\mathrm{Pf}\left(\frac{1}{|x|}\right) - \gamma\delta_0, \tag{57}$$

where γ is Euler's constant: cf. e.g. [Schw][3]. Therefore

$$\begin{aligned}\mathfrak{v}_{\infty,\lambda} &= C_0\,(\log\lambda)\ \delta - C_0\mathcal{F}^{-1}(\log|\zeta|) + C_1\delta \\ &= \mathcal{F}^{-1}\left(-C_0\log(|\zeta|/\lambda) + C_1\right),\end{aligned}$$

and

$$\mathcal{F}\left(R_{00}(-\beta^2)^{1/2}\,(\lambda\mathfrak{v}(\lambda\cdot) - \mathfrak{v}_{\infty,\lambda})\,R_{00}(-\beta^2)^{1/2}\right)\mathcal{F}^{-1}$$

is an integral operator with kernel:

$$\frac{1}{2\pi}\frac{1}{(\zeta^2/2+\beta^2)^{1/2}}\ \mathfrak{e}\left(\frac{\zeta-\zeta'}{\lambda}\right)\frac{1}{(\zeta'^2/2+\beta^2)^{1/2}}, \tag{58}$$

since multiplication by a distribution $a(x)$ becomes an integral operator with kernel $(2\pi)^{-1}\mathcal{F}a(\zeta-\zeta')$ after conjugation by $\mathcal{F}$. Since conjugation by the Fourier transform does not change the operator norm, it follows that the norm in (56) can be bounded by the Hilbert-Schmidt norm of (58), whose square equals:

$$\begin{aligned}&\frac{1}{\pi^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{((\zeta-\eta)^2+2\beta^2)(\zeta^2+2\beta^2)}d\zeta\right)|\mathfrak{e}\left(\frac{\eta}{\lambda}\right)|^2 d\eta \\ =\ &\frac{\sqrt{2}}{\pi\beta}\int_{\mathbb{R}}\frac{|\mathfrak{e}(\eta/\lambda)|^2}{\eta^2+8\beta^2}d\eta = \frac{\sqrt{2}}{\pi\beta\lambda}\int_{\mathbb{R}}\frac{|\mathfrak{e}(\eta)|^2}{\eta^2+(\beta^2/8\lambda^2)}d\eta \le \frac{C_{\mathfrak{v}}^2\sqrt{2}}{\pi\beta\lambda}.\end{aligned}$$

Here we have used the elementary integral identity:

$$\int_{\mathbb{R}}\frac{1}{(a\zeta^2+b)(a(\zeta-\eta)^2+b)}d\zeta = \frac{2\pi}{\sqrt{ab}}\frac{1}{a\eta^2+4b}\ , \tag{59}$$

where $a, b > 0$. This finishes the proof of lemma 4.1. QED.

We will apply the previous lemma to our potentials (10), but before doing so we first state and prove a weaker variant, which will be used to prove Theorem 1.1. Let us introduce the (numerical) constant:

$$C_{(60)} := \left(\frac{1}{4\pi}\int_{\mathbb{R}}\frac{(|\log|\eta|\,|+2)^2}{\eta^2+4}\,d\eta\right)^{1/2}. \tag{60}$$

Numerical evaluation of the integral (using either Mathematica or Maple 8) gives $C_{(60)}^2 \simeq 1.53$.

Lemma 4.2. *Let $\mathfrak{v} = \mathfrak{v}(z)$ be an $L(F)$-valued tempered distribution on $\mathbb{R}$ such that*

$$C_{\mathfrak{v}} := \sup_{\zeta\in\mathbb{R}}\|\ (|\log|\zeta|\,|+1)^{-1}\ \mathcal{F}\mathfrak{v}(\zeta)\ \| < \infty. \tag{61}$$

Then for all $\lambda \ge e$ and all $\varepsilon > 0$,

$$\|\ (-\varepsilon\Delta+\varepsilon^{-1})^{-1/2}\left(\frac{\lambda}{\log\lambda}\mathfrak{v}(\lambda z)\right)(-\varepsilon\Delta+\varepsilon^{-1})^{-1/2}\ \| \le C_{(60)}C_{\mathfrak{v}}\,(|\log\varepsilon|+2)\,. \tag{62}$$

Proof. Conjugating as before by the Fourier transform, and estimating the operator norm by the Hilbert-Schmidt one, we find that the left hand side of (62) is bounded, by the square root of

$$(2\pi)^{-2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{(\varepsilon\zeta^2+\varepsilon^{-1})(\varepsilon(\zeta-\eta)^2+\varepsilon^{-1})}d\zeta\right)\frac{1}{(\log\lambda)^2}\ \|\ \mathfrak{v}(\frac{\eta}{\lambda})\ \|^2\ d\eta.$$

[3](57) can also easily be shown directly, using the observation that Pf $(1/|x|)$ and $-2\mathcal{F}^{-1}(\log|\zeta|)$ are both solutions of the distributional equation $x\Lambda = -\mathrm{sgn}\ x$ and therefore only differ by a multiple of δ, which can then be computed to be -2γ.

By (61) we can bound

$$\begin{aligned}\frac{1}{(\log\lambda)^2}\parallel \mathfrak{v}(\eta/\lambda)\parallel^2 &\le C_{\mathfrak{v}}^2\left(\frac{|\log|\eta||}{\log\lambda}+1+\frac{1}{\log\lambda}\right)^2\\ &\le C_{\mathfrak{v}}^2\left(|\log|\eta||+2\right)^2,\end{aligned}$$

since we suppose that $\lambda \ge e$. Hence, using (59) again, we find that our norm is bounded by the square root of

$$\begin{aligned}&\frac{C_{\mathfrak{v}}^2}{2\pi}\int_{\mathbb{R}}\frac{(\,|\log|\eta||+2\,)^2}{\varepsilon\eta^2+4\varepsilon^{-1}}\,d\eta \le \frac{C_{\mathfrak{v}}^2}{2\pi}\int_{\mathbb{R}}\frac{(|\log|\eta||+|\log\varepsilon|+2)^2}{\eta^2+4}d\eta\\ \le\ &\frac{C_{\mathfrak{v}}^2}{\pi}(\log\varepsilon)^2\int_{\mathbb{R}}\frac{d\eta}{\eta^2+4}\ +\ \frac{C_{\mathfrak{v}}^2}{\pi}\int_{\mathbb{R}}\frac{(|\log|\eta||+2)^2}{\eta^2+4}d\eta = C_{\mathfrak{v}}^2\left(\frac{(\log\varepsilon)^2}{2}+4C_{(60)}^2\right),\end{aligned}$$

by (60). Since $C_{(60)}^2 \ge 1/2$, we see that (62) will be bounded by $C_{(60)}C_{\mathfrak{v}}\,(|\log\varepsilon|+2)$, as claimed. QED

The next step will be to apply lemma 4.1 to the potentials V_j^1 and V_{jk}^1, with $\lambda = \sqrt{B}$. We introduce the B-dependent tempered distribution $q = q^B$, and linear operators $C_j^n, C_{jk}^e \in L(F_{\mathbb{M}}^1)$ by:

$$q^B(z) = \log B\ \delta(z) + \mathrm{Pf}\ \left(\frac{1}{|z|}\right), \tag{63}$$

$$C_j^n := C_j^{n,\mathbb{M}} := -\Pi_{\text{eff}}^1 \log\left(\tfrac{1}{4}\left(x_j^2+y_j^2\right)\right)\Pi_{\text{eff}}^1, \tag{64}$$

$$C_{jk}^e := C_{jk}^{e,\mathbb{M}} := -\Pi_{\text{eff}}^1 \log\left(\tfrac{1}{4}\left((x_j-x_k)^2+(y_j-y_k)^2\right)\right)\Pi_{\text{eff}}^1; \tag{65}$$

Observe that (64) and (65) are related to (12) and (13) by conjugation by U_{xy}^B. See also remark 1.4 for a physical interpretation of these three terms.

Lemma 4.3. *Let $R_{00}(-\beta^2) = (-\frac{1}{2}\Delta_z+\beta^2)^{-1}$, $\beta > 0$ be the free resolvent in dimension 1. There exists a positive constant $C_{(66)} := C_{(66)}(\mathbb{M}) > 0$ only depending on $\mathbb{M}$, such that for all $B, \beta > 0$,*

$$\|R_{00}(-\beta^2)^{1/2}\left(\sqrt{B}V_j^1(\sqrt{B}z) - \left(q^B(z)+C_j^n\ \delta(z)\right)\right)R_{00}(-\beta^2)^{1/2}\| \le \frac{C_{(66)}}{\sqrt{\beta}B^{1/4}}, \tag{66}$$

and

$$\|R_{00}(-\beta^2)^{1/2}\left(\sqrt{B}V_{jk}^1(\sqrt{B}z) - \left(q^B(z)+C_{jk}^e\ \delta(z)\right)\right)R_{00}(-\beta^2)^{1/2}\| \le \frac{C_{(66)}}{\sqrt{\beta}B^{1/4}}, \tag{67}$$

the norm being the operator norm on $L^2(\mathbb{R}, F_{\mathbb{M}}^1)$.

To simplify future estimates, we have taken the same constant in both inequalities.

Proof. Recall the formulas (48) and (49) for $V_j^1(z)$ and $V_{jk}^1(z)$. We need the asymptotics of their Fourier-transforms at 0. By [AS, 9.6.21], the Fourier transform of $(1+z^2)^{-1/2}$ equals

$$\mathcal{F}\left((1+z^2)^{-1/2}\right)(\zeta) = 2K_0(|\zeta|),$$

where K_0 is the Macdonald function. Since the projector $\Pi^1_{\rm eff}$ effectively only acts in the x and y-variables, it follows that

$$\begin{aligned} \mathcal{F}V_j^1(\zeta) &= \Pi^1_{\rm eff}\mathcal{F}_{z\to\zeta}\left((x_j^2+y_j^2)^{-1/2}\left(1+((x_j^2+y_j^2)^{-1/2}z)^2\right)^{-1/2}\right)\Pi^1_{\rm eff} \\ &= 2\Pi^1_{\rm eff}K_0\left(\sqrt{x_j^2+y_j^2}\cdot\zeta\right)\Pi^1_{\rm eff}, \end{aligned}$$

with a similar formula for $\mathcal{F}V^1_{jk}$.

Now it is known that

$$K_0(|\zeta|) = -\log|\zeta| + \log 2 - \gamma + O(||\zeta^2\log|\zeta|\ |),\quad |\zeta|\to 0,$$

and that $K_0(|\zeta|)$ is bounded on $|\zeta|\geq 1$ (even exponentially decreasing there): see e.g. [AS, 9.6.13]. It then easily follows that, as $\zeta\to 0$ and as operators on Ran $\Pi^1_{\rm eff}$,

$$\begin{aligned} \mathcal{F}V_j^1(\zeta) &\simeq -2\log|\zeta| - 2\gamma + C_j^n, \\ \mathcal{F}V_{jk}^1(\zeta) &\simeq -2\log|\zeta| - 2\gamma + C_{jk}^e, \end{aligned} \tag{68}$$

with an error of $O(|\zeta^2\log|\zeta|\ |)$. An appeal to lemma 4.1, with $\lambda=\sqrt{B}$ and with $C_0=2$ and $C_1=-2\gamma+C_j^n$ respectively $C_1=-2\gamma+C^e_{jk}$, then finishes the proof. QED

Remark 4.4. An explicit computation of the matrices of C_j^n and C^e_{jk} with respect to the natural basis $X^1_m, m\in\Sigma(\mathbb{M})$ shows that C_j^n and C^e_{jk} do depend on their indices j and j,k, respectively.

We will likewise need lemma 4.2 for $\mathfrak{v}=V_j^1$. We can without loss of generality assume that $j=1$, by permutational symmetry of $\Sigma(\mathbb{M})$. As we have seen above, $\mathcal{F}V_1^1(\zeta) = 2\Pi^1_{\rm eff}K_0(|\zeta|\sqrt{x_1^2+y_1^2})\Pi^1_{\rm eff}$. It can easily be verified that $\|\ (|\log|\zeta|\ |+1)^{-1}K_0(\zeta)\ \|_\infty = 1$, so that, for example,

$$C_{V_1^1}\leq C_{(69)} := 2+2\|\ \Pi^1_{\rm eff}\left|\log\sqrt{x_1^2+y_1^2}\right|\Pi^1_{\rm eff}\ \|. \tag{69}$$

The operator norm on the right and side can be evaluated explicitly, and behaves asymptotically for large positive $\mathbb{M}$ as $2\log(\mathbb{M})$.

We next extend lemma 4.3 to multi-particle potentials. Let us define the multi-particle potential v_C by

$$\begin{aligned} v_C(z) = v_C^B(z) &= -Z\sum_j\left(q^B(z_j)+C_j^n\delta(z_j)\right) \\ &+ \sum_{j<k}\left(q^B(z_j-z_k)+C^e_{jk}\delta(z_j-z_k)\right). \end{aligned} \tag{70}$$

Lemma 4.5. *Let $R_{00}(-\alpha^2) = (-\frac{1}{2}\Delta_z+\alpha^2)^{-1}$, $\alpha>0$, be the resolvent of the free Hamiltonian in $\mathbb{R}^N$. Then*

$$\|R_{00}(-\alpha^2)^{1/2}\left(\sqrt{B}\mathcal{V}^1_{\rm eff}(\sqrt{B}z)-v_C(z)\right)R_{00}(-\alpha^2)^{1/2}\|\leq\frac{C_{(72)}}{\sqrt{\alpha}B^{1/4}}, \tag{71}$$

where

$$C_{(72)} := C_{(72)}(N,Z,\mathbb{M}) := C_{(66)}\ N^{1/4}\left(Z+\frac{1}{2}(N-1)\right). \tag{72}$$

Proof. We split both potentials into their 'electron-nucleus' and 'electron-electron' parts:

$$\mathcal{V}^1_{\rm eff} = \mathcal{V}^1_{{\rm eff},n}+\mathcal{V}^1_{{\rm eff},e},$$

and similarly for v_C: $v_C = v_{C,n} + v_{C,e} = v^B_{C,n} + v^B_{C,e}$. Writing $\mathcal{V}^B_{s,\nu}(z)$ for $\sqrt{B}\mathcal{V}^1_{s,\nu}(\sqrt{B}z)$ (with a mild abuse of notation), where $\nu = n$ or e, we bound the left hand side of (71) by

(73)
$$\|R_{00}(-\alpha^2)^{1/2}\left(\mathcal{V}^B_{\mathrm{eff},n} - v_{C,n}\right)R_{00}(-\alpha^2)^{1/2}\| + \|R_{00}(-\alpha^2)^{1/2}\left(\mathcal{V}^B_{\mathrm{eff},e} - v_{C,e}\right)R_{00}(-\alpha^2)^{1/2}\|,$$

and estimate the two terms separately. Let $R_{00,j}(-\beta^2)$ be the 1-dimensional resolvent in the variable z_j, with a β which will be picked below. We will simply write R_{00} for $R_{00}(-\alpha^2)$ and $R_{00,j}$ for $R_{00,j}(-\beta^2)$. If we put

$$\Delta V_j := V^B_j(z_j) - q^B(z_j) - C^n_j\delta(z_j),$$

and

$$\Delta\mathcal{V}_n := Z\sum_j \Delta V_j = \mathcal{V}^B_{\mathrm{eff},n} - v_{C,n}\ ,$$

then, by (66),

$$\begin{aligned}
R_{00}^{1/2}\Delta\mathcal{V}_n R_{00}^{1/2} &= Z\sum_j (R_{00}^{1/2}R_{00,j}^{-1/2})\ (R_{00,j}^{1/2}\Delta V_j R_{00,j}^{1/2})\ (R_{00,j}^{-1/2}R_{00}^{1/2}) \\
&\le \frac{C_{(66)}Z}{\sqrt{\beta}B^{1/4}}\sum_j R_{00}^{1/2}R_{00,j}^{-1}R_{00}^{1/2} = \frac{C_{(66)}Z}{\sqrt{\beta}B^{1/4}}R_{00}(-\alpha)^2(-\frac{1}{2}\Delta_z + N\beta^2) \\
&\le \frac{C_{(66)}Z}{\sqrt{\beta}B^{1/4}}\max_{\xi\in\mathbb{R}^N}\frac{|\xi|^2/2 + N\beta^2}{|\xi|^2/2+\alpha^2} = \frac{C_{(66)}Z\max(1, N\beta^2/\alpha^2)}{\sqrt{\beta}B^{1/4}} = \frac{C_{(66)}ZN^{1/4}}{\sqrt{\alpha}B^{1/4}}
\end{aligned}$$

if we pick $\beta = \alpha/\sqrt{N}$; this choice actually minimizes $\beta^{-1/2}\max(1, N\beta^2/\alpha^2)$ as a function of $\beta \ge 0$, as is easily checked. Similar estimates show that $-R_{00}^{1/2}\Delta\mathcal{V}_n R_{00}^{1/2}$ is bounded from above, in operator sense, by the same number, and we therefore conclude that the first norm in (73) is bounded by $C_{(66)}ZN^{1/4}/\sqrt{\alpha}B^{1/4}$.

To estimate the second term of (73), we will use the following lemma, which is analogous to lemma 3.2 from section 2. Let $\Delta_j = -d^2/dz_j^2$, $\Delta_k = -d^2/dz_k^2$.

Lemma 4.6. *Let $\mathfrak{v} = \mathfrak{v}(z)$ be an $L(F)$-valued distribution on $\mathbb{R}$ (F a finite-dimensional Hilbert space), such that $R_{00}(-\beta^2)^{1/2}\mathfrak{v}R_{00}(-\beta^2)^{1/2}$ is self-adjoint, for $\beta > 0$. Then for all $\mu > 0$,*

(74)
$$\begin{aligned}
&\|\ (-\frac{1}{2}\Delta_j - \frac{1}{2}\Delta_k + \mu^2)^{-1/2}\ \mathfrak{v}(z_j - z_k)\ (-\frac{1}{2}\Delta_j - \frac{1}{2}\Delta_k + \mu^2)^{-1/2}\ \| \\
&\le \frac{1}{2}\|\ (-\frac{1}{2}\Delta_s + \frac{\mu^2}{2})^{-1/2}\ \mathfrak{v}(s)\ (-\frac{1}{2}\Delta_s + \frac{\mu^2}{2})^{-1/2}\ \|,
\end{aligned}$$

where the norm on the left hand side is of course taken in $L^2(\mathbb{R}^2, F)$.

Proof. We use a similar change of variables as in the proof of lemma 3.2: $s = (z_j - z_k)/\sqrt{2}$, $t = (z_j + z_k)/\sqrt{2}$. Then, with $\simeq$ denoting unitary equivalence,

$$\begin{aligned}
&\left(-\frac{1}{2}\Delta_j - \frac{1}{2}\Delta_k + \mu^2\right)^{-1/2}\ \mathfrak{v}(z_j - z_k)\ \left(-\frac{1}{2}\Delta_j - \frac{1}{2}\Delta_k + \mu^2\right)^{-1/2} \\
\simeq\ &\left(-\frac{1}{2}\Delta_s - \frac{1}{2}\Delta_t + \mu^2\right)^{-1/2}\ \mathfrak{v}(\sqrt{2}s)\ \left(-\frac{1}{2}\Delta_s - \frac{1}{2}\Delta_t + \mu^2\right)^{-1/2} \\
\simeq\ &\frac{1}{2}\left(-\frac{1}{2}\Delta_s - \frac{1}{2}\Delta_t + \frac{\mu^2}{2}\right)^{-1/2}\ \mathfrak{v}(s)\ \left(-\frac{1}{2}\Delta_s - \frac{1}{2}\Delta_t + \frac{\mu^2}{2}\right)^{-1/2}.
\end{aligned}$$

Observing that

$$\|\ \left(-\frac{1}{2}\Delta_s + \frac{\mu^2}{2}\right)^{1/2}\left(-\frac{1}{2}\Delta_s - \frac{1}{2}\Delta_t + \frac{\mu^2}{2}\right)^{-1/2}\ \| \le 1.$$

on $L^2(\mathbb{R}^2)$, the lemma follows. QED

Let us write

$$R_{00,jk} = R_{00,jk}(-\mu^2) = \left(-\frac{1}{2}(\Delta_j + \Delta_k) + \mu^2\right)^{-1/2}, \tag{75}$$

the 2-dimensional free resolvent, where μ will be optimized at the end of the proof. Recall that $R_{00} = R_{00}(-\alpha^2)$, and put

$$\Delta V_{jk} = V_{jk}^B(z_j - z_k) - q^B(z_j - z_k) - C_{jk}^e \delta(z_j - z_k).$$

Then, using lemmas 4.6 and 4.3, $R^{1/2}(\mathcal{V}_{\text{eff},e}^B - v_{C,e}^B)R_{00}^{1/2}$ can be estimated from above as follows:

$$\begin{aligned}
R_{00}^{1/2} \sum_{j<k} \Delta V_{jk}\, R_{00}^{1/2} &= \sum_{j<k} \left(R_{00}^{1/2} R_{00,jk}^{-1/2}\right) \left(R_{00,jk}^{1/2}\, \Delta V_{jk}\, R_{00,jk}^{1/2}\right) \left(R_{00,jk}^{-1/2} R_{00}^{1/2}\right) \\
&\leq \frac{C_{(66)}}{2^{3/4}\sqrt{\mu} B^{1/4}} \sum_{j<k} R_{00}^{1/2} R_{00,jk}^{-1} R_{00}^{1/2} \\
&= \frac{C_{(66)}}{2^{3/4}\sqrt{\mu} B^{1/4}} R_{00}(-\alpha^2) \left(\sum_{j<k} (-\frac{1}{2}(\Delta_j + \Delta_k) + \mu^2)\right) \\
&= \frac{C_{(66)}(N-1)}{2^{3/4}\sqrt{\mu} B^{1/4}} R_{00}(-\alpha^2) \left(-\frac{1}{2}\Delta_z + \frac{N\mu^2}{2}\right) \\
&\leq \frac{C_{(66)}(N-1)}{2^{3/4} B^{1/4}} \cdot \frac{1}{\sqrt{\mu}} \max\left(1, \frac{N\mu^2}{2\alpha^2}\right) \leq \frac{C_{(66)}(N-1)N^{1/4}}{2\sqrt{\alpha} B^{1/4}},
\end{aligned}$$

where we minimized the right hand side over μ by choosing $\mu = \alpha\sqrt{2/N}$. The similar upper bound for$-R^{1/2}(\mathcal{V}_e - v_{C,e}^B)R_{00}^{1/2}$ gives the desired estimate for the second norm in (73), and combining the two estimates, we have proved lemma 4.5. QED

We now derive a similar estimate for $\mathcal{W}^B$ as $B \to \infty$.

Lemma 4.7. *Let $R_{00}(-\alpha^2)$ be the free resolvent in dimension N, and let $U = U_{xy}^B$ be the unitary transformation defined by (46). Then, if $\xi \leq 0$,*

$$||R_{00}(-\alpha^2)^{1/2} U^* \mathcal{W}^B U R_{00}(-\alpha^2)^{1/2}|| \leq \frac{C_{(77)}}{\alpha\sqrt{B}}, \tag{76}$$

with

$$C_{(77)} := C_{(77)}(N, Z) := 2\pi^{3/2} N^{3/2} \left(Z^2 + \frac{(N-1)^2}{4}\right). \tag{77}$$

Proof. Recall that $\mathcal{W}^B = -\mathcal{V}_{\text{eff},\perp} R \mathcal{V}_{\perp,\text{eff}}$, where $R = R(\xi) = (H_\perp - \xi)^{-1}$ on $\operatorname{Ran} \Pi_\perp^B$. Hence

$$U^* \mathcal{W}^B U = -B \mathcal{V}_{\text{eff},\perp}^1(\sqrt{B}z) U^* R U \mathcal{V}_{\perp,\text{eff}}^1(\sqrt{B}z),$$

where $\mathcal{V}_{\text{eff},\perp}^1 = \Pi_{\text{eff}}^1 \mathcal{V} \Pi_\perp^1$, and similarly for $\mathcal{V}_{\perp,\text{eff}}^1$. Since for $\xi \leq 0$, $0 \leq R \leq 2/B$ (see section 3), we have that, letting $\mathcal{V}(\cdot, z)$ be the function $(x, y) \to \mathcal{V}(x, y, z)$,

$$\begin{aligned}
0 \leq -U^* \mathcal{W}^B U &\leq 2\Pi_{\text{eff}}^1 \mathcal{V}(\cdot, \sqrt{B}z) \Pi_\perp \mathcal{V}(\cdot, \sqrt{B}z) \Pi_{\text{eff}}^1 \\
&\leq 2\Pi_{\text{eff}}^1 \mathcal{V}(\cdot, \sqrt{B}z)^2 \Pi_{\text{eff}}^1 \\
&\leq 4\left(\Pi_{\text{eff}}^1 \mathcal{V}_n(\cdot, \sqrt{B}z)^2 \Pi_{\text{eff}}^1 + \Pi_{\text{eff}}^1 \mathcal{V}_e(\cdot, \sqrt{B}z)^2 \Pi_{\text{eff}}^1\right).
\end{aligned} \tag{78}$$

As in the previous lemma, we treat the two terms separately. By Cauchy-Schwarz,

$$\mathcal{V}_n(x,y,\sqrt{B}z)^2 \leq Z^2 N \sum_{j=1}^{N} \frac{1}{\rho_j^2 + Bz_j^2},$$

where $\rho_j^2 = x_j^2 + y_j^2$. As before, let $R_{00,j}(-\beta^2)$ be the resolvent of $h_{00,j} = -(1/2)d^2/dz_j^2$ on $\mathbb{R}$. We first estimate the L^2-norm of each

$$R_{00,j}(-\beta^2)^{1/2}\Pi^1_{\mathrm{eff}}(\rho_j^2 + Bz_j^2)^{-1}\Pi^1_{\mathrm{eff}}R_{00,j}(-\beta^2)^{1/2}, \tag{79}$$

by conjugating with the Fourier transform $\mathcal{F}$. Since $\mathcal{F}\left((1+z^2)^{-1}\right)(\zeta) = \pi e^{-|\zeta|}$, (79) then becomes an integral operator with kernel

$$\frac{1}{2}B^{-1/2}\left(\tfrac{\zeta_j^2}{2} + \beta^2\right)^{-1/2}\left(\Pi^1_{\mathrm{eff}}\ \rho_j^{-1}e^{-\rho_j B^{-1/2}|\zeta_j - \eta_j|}\ \Pi^1_{\mathrm{eff}}\right)\left(\tfrac{\eta_j^2}{2} + \beta^2\right)^{-1/2}. \tag{80}$$

The norm of (80) can be estimated by its Hilbert-Schmidt norm, whose square can be bounded by:

$$\frac{1}{4B}\|\Pi^1_{\mathrm{eff}}\frac{1}{\rho_j}\Pi^1_{\mathrm{eff}}\|^2\left(\int_{\mathbb{R}}(\tfrac{\xi^2}{2} + \beta^2)^{-1}d\xi\right)^2 = \frac{\pi^2 C^2_{(81)}}{2\beta^2 B},$$

where we have put

$$C_{(81)} := \|\ \Pi^1_{\mathrm{eff}}\rho_j^{-1}\Pi^1_{\mathrm{eff}}\ \|. \tag{81}$$

Note that $C_{(81)}$ is independent of j, because of the permutational symmetry of $\Sigma(\mathbb{M})$. It follows that:

$$R_{00,j}(-\beta^2)^{1/2}\ \Pi^1_{\mathrm{eff}}(\rho_j^2 + Bz_j^2)^{-1}\Pi^1_{\mathrm{eff}}\ R_{00,j}(-\beta^2)^{1/2} \leq \frac{\pi C_{(81)}}{\sqrt{2}\beta\sqrt{B}}, \tag{82}$$

and therefore

$$\begin{aligned} 0 &\leq R_{00}(-\alpha^2)^{1/2}\ \Pi^1_{\mathrm{eff}}\ \mathcal{V}_n(\cdot,\sqrt{B})^2\ \Pi^1_{\mathrm{eff}}\ R_{00}(-\alpha^2)^{1/2} \\ &\leq \frac{\pi C_{(81)}Z^2N}{\sqrt{2}\beta B^{1/2}}R_{00}(-\alpha^2)\sum_j\left(h_{00,j} + \beta^2\right) = \frac{\pi C_{(81)}Z^2N}{\sqrt{2}\beta B^{1/2}}R_{00}(-\alpha^2)(h_{00} + N\beta^2) \\ &\leq \frac{\pi C_{(81)}Z^2N^{3/2}}{\sqrt{2}\alpha B^{1/2}}, \end{aligned} \tag{83}$$

if we choose $\beta = \alpha/\sqrt{N}$. The same inequality then holds for the norm, since the operator we estimate is positive.

We next treat the interaction term $R_{00}(\alpha^2)^{1/2}\mathcal{V}_e(\cdot,\sqrt{B}z)^2R_{00}(\alpha^2)^{1/2}$ in a similar way as in the proof of lemma 4.5. First, by Cauchy-Schwarz again,

$$\mathcal{V}_e(\cdot,\sqrt{B}z)^2 \leq \frac{N(N-1)}{2}\sum_{j<k}\frac{1}{\rho_{jk}^2 + B(z_j - z_k)^2},$$

where we have put $\rho_{jk}^2 = (x_j - x_k)^2 + (y_j - y_k)^2$. Since the rotation $(r_j, r_k) \to \left(2^{-1/2}(r_j - r_k), 2^{-1/2}(r_j + r_k)\right)$ commutes with Π^1_{eff} (since it commutes with H^1_0 and with $\mathbb{L}_z$), we find, after a unitary transformation, that

$$\begin{aligned} &\|\ R_{00,jk}(-\mu^2)^{1/2}\Pi^1_{\mathrm{eff}}\left(\rho_{jk}^2 + B(z_j - z_k)^2\right)^{-1}\Pi^1_{\mathrm{eff}}R_{00,jk}(-\mu^2)^{1/2}\ \| \\ \leq\ &\|\ \frac{1}{2}R_{00,j}(-\frac{\mu^2}{2})^{1/2}\Pi^1_{\mathrm{eff}}\left(\rho_j^2 + Bz_j^2\right)^{-1}\Pi^1_{\mathrm{eff}}R_{00,j}(-\frac{\mu^2}{2})^{1/2}\ \| \\ \leq\ &\frac{\pi C_{(81)}}{2\mu\sqrt{B}}; \end{aligned}$$

compare the proof of lemma 4.6. Hence, using similar arguments as before,

$$\begin{aligned} & R_{00}(-\alpha^2)^{1/2}\,\Pi^1_{\rm eff}\,\mathcal{V}_e(\cdot,\sqrt{B})^2\,\Pi^1_{\rm eff}\,R_{00}(-\alpha^2)^{1/2} \\ \leq\ & \frac{\pi C_{(81)}N(N-1)}{4\mu\sqrt{B}}\,R_{00}(-\alpha^2)\sum_{j<k}(h_{00,jk}+\mu^2) \\ \leq\ & \frac{\pi C_{(81)}N(N-1)^2}{4\mu\sqrt{B}}R_{00}(-\alpha^2)(h_{00}+\frac{N\mu^2}{2}) \\ \leq\ & \frac{\pi C_{(81)}N^{3/2}(N-1)^2}{4\sqrt{2}\alpha\sqrt{B}}, \end{aligned}$$

if we choose $\mu=\alpha\sqrt{2/N}$. Adding this estimate to the one for $\mathcal{V}_n^2$, and remembering the factor 4 from (78), we have proved (76) with the constant $C_{(77)}=2^{3/2}\pi C_{(81)}N^{3/2}\left(Z^2+\frac{(N-1)^2}{4}\right)$. A priori, $C_{(81)}$ might still depend on $\mathbb{M}$, but it in fact does not, as we will finally show. We compute $C_{(81)}$ in the Landau basis (28) of $F_{\mathbb{M}}{}^1$, with respect to which $\Pi^1_{\rm eff}\,\rho_1^{-1}\,\Pi^1_{\rm eff}$ diagonalizes:

$$\begin{aligned} C_{(81)} &= \max_{0\leq m\leq\mathbb{M}}\frac{1}{2^m m!}\int_0^\infty \rho_1^{2m}e^{-\rho_1^2/2}d\rho_1 \\ &= \max_{0\leq m\leq\mathbb{M}}\frac{1}{m!\sqrt{2}}\int_0^\infty s^{m-\frac12}e^{-s}ds \\ &= \max_{0\leq m\leq\mathbb{M}}\frac{\Gamma(m+\frac12)}{\sqrt{2}\Gamma(m+1)}. \end{aligned}$$

It is known that

$$\Gamma\left(m+\frac12\right)=\frac{1\cdot3\cdot5\cdot\cdots\cdot(2m-1)}{2^m}\Gamma\left(\frac12\right),$$

cf. e.g. [AS], formula (6.1.12), page 255. Using this, one easily finds that

$$C_{(81)}=\frac{1}{\sqrt{2}}\Gamma\left(\frac12\right)=\sqrt{\frac{\pi}{2}},$$

which completes the proof of the lemma. QED

To prove Theorem 1.5 we will need to control the Sobolev norm of $V_1^B-v_\delta^B$. This is done in the following lemma, which we formulate in slightly greater generality than needed, with an eye to future applications.

Lemma 4.8. *Let $V_1^B:=V_1^{B,\mathbb{M}}$ be defined as in (10), $c>0$ and let $\alpha_c=\alpha_c(B)$ be the unique positive solution of*

$$\alpha_c=\frac{2}{c}\log\left(\frac{\sqrt{B}}{\alpha_c}\right), \tag{84}$$

Then the constant
(85)

$$C_{(85)}:=\left(\pi^2+9\log^2(2)+\frac{64\sqrt{2}}{\pi}+\sup_{|\zeta|\leq1}\left|\mathcal{F}V_1^1(\zeta)+2\log(|\zeta|)\right|^2+\frac{8\sqrt{2}}{\pi}\sup_{|\zeta|\geq1}|\mathcal{F}V_1^1(\zeta)|^2\right)^{1/2}$$

is finite and depends only on $\mathbb{M}$. Moreover for all $c>0$ and all $B>0$

$$\|R_{00}(-\alpha_c^2)^{\frac12}(V_1^B-c\alpha_c(B)\delta)R_{00}(-\alpha_c^2)^{\frac12}\|\leq\frac{C_{(85)}}{\alpha_c(B)}. \tag{86}$$

Observe that if $c = 2$, then $\alpha_c(B)$ is the $\alpha(B)$ from Theorem 1.1: cf. (6). Also note that the constant $C_{(85)}$ is independent of $c > 0$.

Proof. The first statement follows at once using the explicit knowledge of $\mathcal{F}V_1^1$ obtained in the proof of lemma 4.3. To prove the estimate (86) we introduce the auxiliary function X by

$$\begin{aligned}\mathcal{F}V_1^B(\zeta) - c\alpha_c\mathcal{F}\delta(\zeta) &= \mathcal{F}V_1^1(\frac{\zeta}{\sqrt{B}}) + 2\log\frac{|\zeta|}{\sqrt{B}} - 2\log\frac{|\zeta|}{\alpha_c} \\ &=: X(\frac{\zeta}{\sqrt{B}}) - 2\log\frac{|\zeta|}{\alpha_c},\end{aligned}$$

where in the first line we used equation (84). The Fourier transform of $V_1^B - c\alpha_c(B)\delta$ acts as convolution by a function, and we estimate the norm of $Y := R_{00}(-\alpha_c^2)^{\frac{1}{2}}(V_1^B - c\alpha_c\delta)R_{00}(-\alpha_c^2)^{\frac{1}{2}}$ by its Hilbert-Schmidt norm, as in the proofs of lemmas 4.1, 4.2. It follows that

$$\|Y\|^2 \le \frac{\sqrt{2}}{\pi\alpha_c}\int_{\mathbb{R}}\frac{|\mathcal{F}V_1^B(\zeta) - c\alpha_c\mathcal{F}\delta(\zeta)|^2}{8\alpha_c^2+\zeta^2}d\zeta \le \frac{2\sqrt{2}}{\pi\alpha_c}\int_{\mathbb{R}}\frac{|X(\frac{\zeta}{\sqrt{B}})|^2 + |2\log\frac{|\zeta|}{\alpha_c}|^2}{8\alpha_c^2+\zeta^2}d\zeta.$$

First,

$$\frac{1}{\alpha_c}\int_{\mathbb{R}}\frac{4}{8\alpha_c^2+\zeta^2}\left|\log\frac{|\zeta|}{\alpha_c}\right|^2 d\zeta = \frac{\pi}{2\sqrt{2}\alpha_c^2}(\pi^2 + 9\log^2(2)).$$

We next look at the contribution of X, which we split in two parts:

$$\begin{aligned}\int_{\sqrt{B}}^{\infty}\frac{|X(\frac{\zeta}{\sqrt{B}})|^2 d\zeta}{8\alpha_c^2+\zeta^2} &= \frac{1}{\sqrt{B}}\int_1^{\infty}\frac{|X(\zeta)|^2 d\zeta}{8B^{-1}\alpha_c^2+\zeta^2} \le \frac{2}{\sqrt{B}}\int_1^{\infty}\frac{|\mathcal{F}V(\zeta)|^2 + 4(\log|\zeta|)^2}{\zeta^2}d\zeta \\ &\le \frac{2}{\sqrt{B}}\left(\sup_{|\zeta|\ge 1}|\mathcal{F}V(\zeta)|^2 \ + \ 8\right),\end{aligned}$$

since

$$\int_1^{\infty}\frac{(\log|\zeta|)^2}{|\zeta|^2}d\zeta = 2,$$

and

$$\begin{aligned}\int_0^{\sqrt{B}}\frac{|X(\frac{\zeta}{\sqrt{B}})|^2 d\zeta}{8\alpha_c^2+\zeta^2} &\le \sup_{|\zeta|\le 1}|X(\zeta)|^2\int_0^{\sqrt{B}}\frac{d\zeta}{8\alpha_c^2+\zeta^2} \\ &\le \frac{\pi}{4\sqrt{2}\alpha_c}\sup_{|\zeta|\le 1}|X(\zeta)|^2.\end{aligned}$$

The rest is now elementary. Notice in particular that $\sup_{B>0}\alpha_c(B)/\sqrt{B} = 1$. QED

The limit potential (70) suggests defining an effective Hamiltonian $h_C = h_C^B$ by:

$$h_C := h_{00} + v_C. \tag{87}$$

As it stands, this is just a formal expression, and our first task is to give a meaning to h_C as a self-adjoint operator on $L^2(\mathbb{R}^N; F_{\mathbb{M}}^1)$. We do this by showing that v_C is form-bounded with respect h_{00}, with zero relative form-bound. Let $\langle\cdot,\cdot\rangle$ denote the duality between distributions and test functions.

Lemma 4.9. *The quadratic forms $u \to \langle\delta, |u|^2\rangle = |u(0)|^2$ and $u \to \langle\mathrm{Pf}(1/|z|), |u|^2\rangle$ are well-defined on $H^1(\mathbb{R})$, and form-bounded with respect to h_{00}, with relative bound zero. More precisely, we have for all $\varepsilon > 0$ that*

$$\delta \le \tfrac{1}{2}(-\varepsilon\Delta_z + \varepsilon^{-1}), \tag{88}$$

and

$$\mathrm{Pf}(|z|^{-1}) \leq C_{(90)}(|\log\varepsilon| + 1)(-\varepsilon\Delta_z + \varepsilon^{-1}), \tag{89}$$

where

$$C_{(90)} := \sqrt{\frac{\pi^2}{2} + 2(\log 2)^2} + \gamma. \tag{90}$$

Proof. This is well-known for δ. For Pf $(|z|^{-1})$ we first note that, since Pf $(|z|^{-1}) = -2\mathcal{F}^{-1}(\log|\zeta|) - 2\gamma\delta_0$, it suffices to prove the form-boundedness of $\mathcal{F}^{-1}(\log|\zeta|)$. The latter will follow from:

$$\|(-\varepsilon\Delta + \varepsilon^{-1})^{-1/2}\mathcal{F}^{-1}(\log|\zeta|)(-\varepsilon\Delta + \varepsilon^{-1})^{-1/2}\|^2 \leq \frac{1}{2}(\log\varepsilon)^2 + \frac{\pi^2 + 4(\log 2)^2}{8}, \tag{91}$$

for all $\varepsilon > 0$. To prove (91), observe that after conjugation by the Fourier transform $\mathcal{F}$, and estimating the operator norm by the Hilbert-Schmidt norm, the square of (91) can be bounded by:

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\log|\zeta - \eta|)^2}{(\varepsilon\zeta^2 + \varepsilon^{-1})(\varepsilon\eta^2 + \varepsilon^{-1})} d\zeta d\eta. \tag{92}$$

Changing variables and using (59), we find that (92) can be bounded by

$$\begin{aligned} & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\log|\zeta|)^2}{\varepsilon\zeta^2 + 4\varepsilon^{-1}} d\zeta = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(\log|\zeta/\varepsilon|)^2}{\zeta^2 + 4} d\xi \\ \leq\ & \frac{1}{\pi}(\log\varepsilon)^2 \int_{\mathbb{R}} \frac{d\zeta}{\zeta^2 + 4} + \frac{1}{\pi}\int_{\mathbb{R}} \frac{(\log\zeta)^2}{\zeta^2 + 4} d\zeta = \frac{1}{2}(\log\varepsilon)^2 + \frac{\pi^2 + 4(\log 2)^2}{8}. \end{aligned}$$

Hence, using (88),

$$\begin{aligned} & \left|\langle \mathrm{Pf}\ (|x|^{-1}), |u|^2\rangle\right| = 2\cdot\ \left|\ \langle \mathcal{F}^{-1}(\log|\xi|), |u|^2\rangle\ +\ \gamma\langle\delta, |u|^2\rangle\ \right| \\ \leq\ & \left\{\left(2(\log\varepsilon)^2 + \frac{\pi^2}{2} + 2(\log 2)^2\right)^{1/2} + \gamma\right\}\left\{-\varepsilon(\Delta u, u) + \varepsilon^{-1}\|u\|^2\right\} \\ =\ & \left(\sqrt{2}|\log\varepsilon| + C_{(90)}\right)\left\{-\varepsilon(\Delta u, u) + \varepsilon^{-1}\|u\|^2\right\}, \end{aligned}$$

which implies (89) since $\sqrt{2}/C_{(90)} < 1$. QED

Because of the terms $C_j^n\delta(z_j)$ and $C_{jk}^e\delta(z_j - z_k)$ we have to extend the first part of lemma 4.9 to the vector-valued case, but this is immediate: we just note that if C a linear operator on $F_{\mathbb{M}}^1$ (or any finite-dimensional vector space, for that matter), then the right interpretation of $C\delta$ as quadratic form on $H^1(\mathbb{R}, F_{\mathbb{M}}^1)$ is given by $\langle\delta(z),\ (Cu(z), u(z))\rangle$, where $(\cdot,\cdot)$ is the inner product on $F_{\mathbb{M}}^1$.

Finally, we lift lemma 4.9 to $\mathbb{R}^N$. Recall that if $L : \mathbb{R}^N \to \mathbb{R}$ is a linear map, then the pull-back $L^*\Lambda$ of a distribution Λ on $\mathbb{R}$ is well-defined, and can be computed by going to linear coordinates z' with respect to which $L(z') = z'_1$. It then immediately follows from lemma 4.9 that $L^*\delta$ and L^*q^B will be form-bounded with respect to h_{00} on $\mathbb{R}^N$, with relative bound 0. Taking $L(z) = L_j(z) = z_j$ and $L(z) = L_{jk}(z) = z_j - z_k$, we see that the sesqui-linear form $t_C^B(u)$ given by

$$\begin{aligned} t_C^B(u) &= \frac{1}{2}||\nabla u||^2 - Z\sum_j \left\langle L_j^*q^B,\ |u|^2\right\rangle + \left\langle (L_j^*\delta,\ (C_j^n u, u)\right\rangle \\ &+ \sum_{j<k}\left\langle L_{jk}^*q^B,\ |u|^2\right\rangle + \left\langle L_{jk}^*\delta,\ (C_{jk}^e u, u)\right\rangle, \end{aligned} \tag{93}$$

is well-defined on $H^1(\mathbb{R}^N, F_{\mathbb{M}}^1)$, and bounded from below by $-C||u||^2$, for some constant C, depending on B, Z, N and $\mathbb{M}$. By the Kato-Lax-Lions-Milgram-Nelson

Theorem (cf. e.g. [RS, Theorem X.17]), t_C^B defines a unique self-adjoint operator, which we will call $h_C = h_C^{B,\mathbb{M}}$, and informally write as (87). In Appendix A we will give a characterization of the operator domain of h_C^B. Similar arguments will define h_δ^B as a self-adjoint operator, cf. [BD].

5. **Proof of Theorem 1.1**

We will first compare the resolvents of $h_{\text{eff}} := h_{\text{eff}}^{B,\mathbb{M}}$ and of $h_{\text{eff}} + \mathcal{W}$, where $\mathcal{W} := \mathcal{W}^{B,\mathbb{M}}(\xi)$ was defined at the beginning of section 3. Put

$$r_{\text{eff}} := r_{\text{eff}}(\xi) := (h_{\text{eff}} - \xi)^{-1}, \quad R_{\text{eff}}^{\mathcal{W}} := R_{\text{eff}}^{\mathcal{W}}(\xi) := (h_{\text{eff}} + \mathcal{W} - \xi)^{-1},$$

and let

$$B_{(94)} := \frac{4C_{(77)}^2}{\alpha\Big(C_{(77)}\Big)^2}, \tag{94}$$

where $\alpha = \alpha(B)$ is the function defined by (6) , and

$$c_{\text{eff}} := 2\left(\frac{N}{2\varepsilon_{\text{eff}}^2} + 1\right) C_{(77)}, \tag{95}$$

with $\varepsilon_{\text{eff}} = \varepsilon_{\text{eff}}(Z, \mathbb{M})$ the unique positive solution of

$$ZC_{(60)}\, C_{V_1^1}\, \varepsilon(|\log\varepsilon| + 2) = \frac{1}{4}, \tag{96}$$

where $C_{V_1^1}$ is defined by (61) with $\mathfrak{v} = V_1^1$ (see (69) for an upper bound). Note that both constants only depend on N, Z and $\mathbb{M}$, and this in a controlled way.

We then have:

Theorem 5.1. *If $B \geq B_{(94)}$, $\xi \leq 0$ and*

$$c_{\text{eff}} \frac{\alpha}{\sqrt{B}} \leq d_{\text{eff}}(\xi) \leq \frac{1}{2}\alpha^2, \tag{97}$$

with $\alpha = \alpha(B)$ is as in Theorem 1.1, then $\xi \in \rho(h_{\text{eff}} + \mathcal{W})$, and $\|R_{\text{eff}}^{\mathcal{W}}(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$. Furthermore,

$$\| R_{\text{eff}}^{\mathcal{W}}(\xi) - r_{\text{eff}}(\xi)\| \leq c_{\text{eff}} \frac{\alpha}{d_{\text{eff}}(\xi)^2\sqrt{B}}. \tag{98}$$

Proof of Theorem 5.1. It clearly suffices to establish Theorem 5.1 after conjugation by U_{xy}^B, defined by (46). To simplify notations, we will simply denote the conjugated operators by the same letters as the original ones. Using the symmetrized resolvent formula we estimate

$$\| R_{\text{eff}}^{\mathcal{W}}(\xi) - r_{\text{eff}}(\xi) \| \leq \frac{1}{d_{\text{eff}}(\xi)} \left(\frac{\|K_{\text{eff}}(\xi)\|}{1 - \|K_{\text{eff}}(\xi)\|}\right), \tag{99}$$

where

$$K_{\text{eff}}(\xi) := |r_{\text{eff}}(\xi)|^{1/2} \mathcal{W}(\xi) r_{\text{eff}}(\xi)^{1/2}, \tag{100}$$

with the convention that $A^{1/2} := \text{sgn}\,(A)|A|^{1/2}$, if A is a self-adjoint operator[4]. Now let $\mu < \inf(\sigma(h_{\text{eff}})$, to be specified later. The following elementary lemma will allow us to replace ξ by μ in (100).

[4]note that since ξ is not necessarily below the infimum of the spectrum, $r_{\text{eff}}(\xi)$ is not necessarily positive, and we use the symmetrized resolvent formula in the following form: $R_{\text{eff}}^{\mathcal{W}} = r_{\text{eff}}^{1/2}(1 + |r_{\text{eff}}|^{1/2}\mathcal{W} r_{\text{eff}}^{1/2})^{-1}|r_{\text{eff}}|^{1/2}$

Lemma 5.2. *If $\mu < \inf \sigma(h_{\mathrm{eff}})$, then for all real ξ in the resolvent set $\rho(h_{\mathrm{eff}})$,*

$$\| \, r_{\mathrm{eff}}(\xi) \, (h_{\mathrm{eff}} - \mu) \, \| \leq \max \left(\frac{|\mu|}{d_{\mathrm{eff}}(\xi)}, 1 \right). \tag{101}$$

Proof. We distinguish two cases: $\inf \sigma(h_{\mathrm{eff}}) < \xi < 0$ and $\xi < \inf \sigma(h_{\mathrm{eff}})$. (Observe that $\inf \sigma(h_{\mathrm{eff}}) < 0$, by the HVZ Theorem, since this is already the case for $N = 1$). In the first case, let (ξ_-, ξ_+) be the largest open interval in $\rho(h_{\mathrm{eff}})$ which contains ξ. Since $[0, \infty)$ is in the spectrum (it is already in the essential spectrum), $\xi_+ \leq 0$. It is easy to see that the function $x \to |x - \mu|/|x - \xi|$ is increasing on $(-\infty, \xi_-) \cap \sigma(h_{\mathrm{eff}})$ and decreasing on $(\xi_+, \infty) \cap \sigma(h_{\mathrm{eff}})$. It follows that

$$\|r_{\mathrm{eff}}(\xi)(h_{\mathrm{eff}} - \mu)\| = \sup_{x \in \sigma(h_{\mathrm{eff}})} \frac{|x - \mu|}{|x - \xi|} = \max \left(\frac{|\xi_- - \mu|}{|\xi_- - \xi|}, \frac{|\xi_+ - \mu|}{|\xi_+ - \xi|} \right) \leq \frac{|\mu|}{d_{\mathrm{eff}}(\xi)},$$

as was to be shown. One shows in a similarly way that (101) equals $(\inf \sigma(h_{\mathrm{eff}}) - \mu)/(\inf \sigma(h_{\mathrm{eff}}) - \xi) \leq |\mu|/d_{\mathrm{eff}}(\xi)$, if $\mu < \xi < \inf \sigma(h_{\mathrm{eff}})$, and is equal to 1 if $\xi < \mu$. QED

Substituting $Id = r_{\mathrm{eff}}(\mu)^{1/2}(h_{\mathrm{eff}} - \mu)^{1/2} = (h_{\mathrm{eff}} - \mu)^{1/2} r_{\mathrm{eff}}(\mu)^{1/2}$ at the appropriate places in formula (100), we see that if $\xi \leq 0$,

$$\|K_{\mathrm{eff}}(\xi)\| \leq \max \left(\frac{|\mu|}{d_{\mathrm{eff}}(\xi)}, 1 \right) \|K_{\mathrm{eff}}(\mu; \xi)\|, \tag{102}$$

with

$$K_{\mathrm{eff}}(\mu; \xi) := r_{\mathrm{eff}}(\mu)^{1/2} \mathcal{W}(\xi) r_{\mathrm{eff}}(\mu)^{1/2}.$$

Repeating the same argument for $K_{\mathrm{eff}}(\mu; \xi)$ using $Id = ((h_{00} + \alpha^2)^{1/2} R_{00}(-\alpha^2)^{1/2}$, we obtain from lemma 4.7 that

$$\|K(\mu; \xi)\| \leq \frac{C_{(77)}}{\alpha \sqrt{B}} \|r_{\mathrm{eff}}(\mu)^{1/2}(h_{00} + \alpha^2)^{1/2}\|^2. \tag{103}$$

We will now estimate the norm on the right hand side, for suitably chosen μ.

Lemma 5.3. *Assume $B \geq e^2$. Define*

$$\mu_{\mathrm{eff}} = \mu_{\mathrm{eff}}(N, Z, \mathbb{M}) := -\frac{\alpha^2}{2} \left(\frac{N}{2\varepsilon_{\mathrm{eff}}^2} + 1 \right), \tag{104}$$

where α is as in Theorem 1.1, and where $\varepsilon = \varepsilon_{\mathrm{eff}}$ is the unique positive solution to the equation (96). Then $\mu_{\mathrm{eff}} < \inf \sigma(h_{\mathrm{eff}})$, and

$$\|r_{\mathrm{eff}}(\mu_{\mathrm{eff}})^{1/2}(h_{00} + \alpha^2)^{1/2}\| \leq \sqrt{2}. \tag{105}$$

Assuming the lemma for the moment, we continue with the proof of Theorem 5.1: we have, by (102), (103) and (105), that if $d_{\mathrm{eff}}(\xi) \geq c_\# \alpha/\sqrt{B}$, then

$$\| \, K_{\mathrm{eff}}(\xi) \, \| \quad \leq \quad 2 \max \left\{ \frac{\alpha^2}{2} \left(\frac{N}{2\varepsilon_{\mathrm{eff}}^2} + 1 \right) \frac{1}{d_{\mathrm{eff}}(\xi)}, \, 1 \right\} \frac{C_{(77)}}{\alpha \sqrt{B}} \leq \frac{1}{2},$$

provided that $c_\#$ satisfies

$$c_\# \geq 2 \left(\frac{N}{2\varepsilon_{\mathrm{eff}}^2} + 1 \right) C_{(77)}, \text{ and } \frac{C_{(77)}}{\alpha \sqrt{B}} \leq \frac{1}{4}.$$

Since $\alpha(B)\sqrt{B}$ is increasing, and since $\alpha(B)\sqrt{B} = x$ iff $B = x^2/(4\alpha(x/4)^2)$, the last inequality is implied by $B \geq B_{(94)}$. Choosing $c_\# = c_{\mathrm{eff}}$ defined by (95), we

conclude that if ξ is such that $d_{\mathrm{eff}}(\xi) \geq c_{\mathrm{eff}}\alpha/\sqrt{B}$, then by (99) , (102), (103) and our choice of $\mu = \mu_{\mathrm{eff}}$,

$$\begin{aligned} \|R^{\mathcal{W}}_{\mathrm{eff}} - r_{\mathrm{eff}}(\xi)\| &\leq \frac{4}{d_{\mathrm{eff}}(\xi)} \max\left\{ \frac{\alpha^2}{2}\left(\frac{N}{2\varepsilon_{\mathrm{eff}}}^{2} + 1\right)\frac{1}{d_{\mathrm{eff}}(\xi)},\ 1 \right\} \frac{C_{(77)}}{\alpha\sqrt{B}} \\ &\leq 2C_{(77)}\left(\frac{N}{2\varepsilon_{\mathrm{eff}}^2} + 1\right)\frac{\alpha}{d_{\mathrm{eff}}(\xi)^2\sqrt{B}}, \end{aligned}$$

provided that $d_{\mathrm{eff}}(\xi) \leq \alpha^2/2$. This proves Theorem 5.1, modulo that of lemma 5.3. QED

Proof of lemma 5.3. We will use a scaling argument. If we let $\simeq$ denote the unitary equivalence induced by the change of variables $z \to z/\alpha$, where $\alpha > 0$ is for the moment a free parameter, and if we write $\mathcal{V}^B_{\mathrm{eff}}$ for $\sqrt{B}\mathcal{V}^1_{\mathrm{eff}}(\sqrt{B}\cdot)$, then if $\mu < \inf\sigma(h_{\mathrm{eff}})$,

$$\begin{aligned} 0 &\leq (h_{00} + \alpha^2)^{1/2} r_{\mathrm{eff}}(\mu)(h_{00} + \alpha^2)^{1/2} \\ &\simeq (h_{00}+1)^{1/2}\left(-\frac{1}{2}\Delta_z + \alpha^{-2}\mathcal{V}^B_{\mathrm{eff}}(\tfrac{\cdot}{\alpha}) - \alpha^{-2}\mu\right)^{-1}(h_{00}+1)^{1/2} \\ &\leq (h_{00}+1)^{1/2}\left(-\frac{1}{2}\Delta_z + \alpha^{-2}\mathcal{V}^B_{\mathrm{eff},n}(\tfrac{\cdot}{\alpha}) - \alpha^{-2}\mu\right)^{-1}(h_{00}+1)^{1/2}, \end{aligned} \tag{106}$$

$\mathcal{V}^1_{\mathrm{eff},n}$ being the attractive part of $\mathcal{V}_{\mathrm{eff}}$. We now choose $\alpha = \log(\sqrt{B}/\alpha)$, as in (6), and put $\lambda := \sqrt{B}/\alpha$. Notice that $B \geq e^2$ implies that $\lambda \geq e$. Then, by lemma 4.2,

$$\begin{aligned} \frac{\sqrt{B}}{\alpha^2}\mathcal{V}^1_{\mathrm{eff},n}\left(\frac{\sqrt{B}}{\alpha}z\right) &= -Z\sum_{j=1}^{N}\frac{\lambda}{\log\lambda}V^1_j(\lambda z_j) \\ &\geq -2C_{(60)}ZC_{V^1_1}\ \varepsilon(|\log\varepsilon| + 2)\left(-\frac{1}{2}\Delta_z + \frac{N}{2\varepsilon^2}\right), \qquad \varepsilon > 0 \\ &= -\frac{1}{2}h_{00} + \frac{N}{4\varepsilon_{\mathrm{eff}}}, \end{aligned}$$

uniformly in λ, if we choose $\varepsilon := \varepsilon_{\mathrm{eff}} = \varepsilon_{\mathrm{eff}}(Z,\mathbb{M})$ such that (96) holds. We now take

$$\mu = \mu_{\mathrm{eff}} = -\frac{\alpha^2}{2}\left(\frac{N}{2\varepsilon_{\mathrm{eff}}^2} + 1\right).$$

Then it follows that $\mu_{\mathrm{eff}} < \inf\sigma(h_{\mathrm{eff}})$, since $h_{\mathrm{eff}} - \mu_{\mathrm{eff}} \geq \frac{1}{2}(h_{00}+1) \geq \frac{1}{2}$. Furthermore, (106) can be estimated by

$$(h_{00}+1)^{1/2}\left(\frac{1}{2}h_{00} - \left(\frac{N}{4\varepsilon_{\mathrm{eff}}^2} + \frac{\mu_{\mathrm{eff}}}{\alpha^2}\right)\right)^{-1}(h_{00}+1)^{1/2} = 2,$$

which implies (105). QED

Proof of Theorem 1.1. It now suffices to combine Theorem 3.1 and Theorem 5.1, while carefully keeping track of the constants. First of all, (97), implies that $\|R^{\mathcal{W}}_{\mathrm{eff}}\| \leq 2\|r_{\mathrm{eff}}(\xi)\|$, and therefore $d^{\mathcal{W}}_{\mathrm{eff}}(\xi) \geq d_{\mathrm{eff}}(\xi)/2$. Furthermore, $\xi < 0$ if $d_{\mathrm{eff}}(\xi) > 0$, and if

$$B \geq B_{\mathrm{eff}} := \max\{B_{(36)}, B_{(94)}, e^2\}, \tag{107}$$

then

$$
\begin{aligned}
& \|(H-\xi)^{-1} - (h_{\text{eff}} - \xi)^{-1} \oplus (H_\perp - \xi)^{-1}\| \\
\leq\ & \|(H-\xi)^{-1} - R^{\mathcal{W}}_{\text{eff}}(\xi) \oplus (H_\perp - \xi)^{-1}\| + \|R^{\mathcal{W}}_{\text{eff}}(\xi) - r_{\text{eff}}(\xi)\| \\
\leq\ & \frac{C_{(37)}}{\sqrt{B}} \frac{1}{d^{\mathcal{W}}_{\text{eff}}(\xi)} + c_{\text{eff}} \frac{\alpha}{d_{\text{eff}}(\xi)^2 \sqrt{B}} \leq \left(2C_{(37)} + c_{\text{eff}} \frac{\alpha}{d_{\text{eff}}(\xi)}\right) \frac{1}{d_{\text{eff}}(\xi)\sqrt{B}} \quad (\text{using } d_{\text{eff}}(\xi)/2 \leq d^{\mathcal{W}}_{\text{eff}}(\xi)\ ) \\
\leq\ & \left(C_{(37)} + \frac{c_{\text{eff}}}{\alpha}\right) \frac{\alpha^2}{d_{\text{eff}}(\xi)^2 \sqrt{B}} \quad (\text{using } d_{\text{eff}}(\xi) \leq \alpha^2/2 \text{ again }) \\
\leq\ & \left(C_{(37)} + \frac{c_{\text{eff}}}{\alpha(B_{\text{eff}})}\right) \frac{\alpha^2}{d_{\text{eff}}(\xi)^2 \sqrt{B}},
\end{aligned}
$$

since $\alpha(B)^{-1}$ is a decreasing function of B. This proves Theorem 1.1 with a constant C_{eff} which is equal to

$$
C_{\text{eff}} := C_{(37)} + \frac{c_{\text{eff}}}{\alpha(B_{\text{eff}})}. \qquad \text{QED}
$$

6. **Proof of Theorem 1.3**

As a first step, we will compare the resolvents $r_{\text{eff}}(\xi)$ of h_{eff} and $r_C(\xi) := (h_C - \xi)^{-1}$ of $h_C := h_C^B$. Recall, that $d_C(\xi) := \text{dist}\,(\xi, \sigma(h_C))$.

Theorem 6.1. *Let $\alpha = \alpha(B)$ be defined by (6). There exist (computable) constants $B'_C, C'_C \geq 0$, only depending on Z, N and $\mathbb{M}$, such that for all $B \geq B'_C$ and all real $\xi \leq 0$ satisfying*

$$
d_C(\xi) \geq C'_C\ \alpha^{3/2} B^{-1/4}, \tag{108}
$$

then $\xi \in \rho(H_{\text{eff}})$, with $\|r_{\text{eff}}(\xi)\| \leq 2\|r_C(\xi)\|$. In addition, letting[5]

$$
C''_C = \max\{C'_C, 4C_{(72)}\left(\frac{N}{2\varepsilon^2_{\text{eff}}} + 1\right)\} \geq C'_C,
$$

then if

$$
d_C(\xi) \geq C''_C\ \alpha^{3/2} B^{-1/4}, \tag{109}
$$

we also have that $\|r_C(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$.

Finally, if

$$
C'_C\ \alpha^{3/2} B^{-1/4} \leq d_C(\xi) \leq \frac{1}{2}\alpha^2, \tag{110}
$$

then

$$
\|r_{\text{eff}}(\xi) - r_C(\xi)\| \leq C'_C \frac{\alpha^{3/2}}{d_C(\xi)^2 B^{1/4}}. \tag{111}
$$

Proof. The proof is similar to the proof of Theorem 5.1, with however some technical changes, due to the fact that v_C is not homogeneous of degree -1, and that its electron-electron part is not positive anymore. As before, we conjugate all operators by U^B_{xy}, keeping the same letters for the conjugated operators.

Arguing as in the proof of Theorem 5.1, one shows that

$$
\|r_{\text{eff}}(\xi) - r_C(\xi)\| \leq \frac{1}{d_C(\xi)} \frac{\|K_C(\xi)\|}{1 - \|K_C(\xi)\|},
$$

[5] with ε_{eff} defined in (96)

where $K_C := |r_C(\xi)|^{1/2}(h_{\text{eff}} - h_C)r_C(\xi)^{1/2}$. Using lemma 4.5, we find that, for any $\mu < \inf \sigma(h_C)$,

$$\|K_C(\xi)\| \le \frac{C_{(72)}}{B^{1/4}\sqrt{\alpha}} \max\left\{\frac{|\mu|}{d_C(\xi)}, 1\right\} \|r_C(\mu)^{1/2}(h_{00} + \alpha^2)^{1/2}\|^2. \tag{112}$$

We then use the following analogue of lemma 5.3, of which we only state a qualitative version.

Lemma 6.2. *There exists a constant $\nu_C = \nu_C(Z, N, \mathbb{M}) \ge 1/2$ such that if $B \ge e$, and if $\mu_C := -\nu_C\alpha^2$, α defined by (6), then $\mu_C < \inf \sigma(h_C)$, and*

$$\|r_C(\mu_C)^{1/2}(h_{00} + \alpha^2)^{1/2}\|^2 \le 2. \tag{113}$$

Proof. As before, we will use scaling. However, contrary to $\delta(z)$, the distribution $\mathrm{Pf}(1/|z|)$ is not homogeneous of degree -1 on $\mathbb{R}$. In fact, if ρ_α is the dilation $\rho_\alpha(z) = \alpha^{-1}z$ on $\mathbb{R}$, $\alpha > 0$ arbitrary, then the pullback of $\mathrm{Pf}\,(1/|\cdot|$ by ρ_α equals

$$\rho_\alpha^*\mathrm{Pf}\left(\frac{1}{|\cdot|}\right) = \alpha\ \mathrm{Pf}\ \left(\frac{1}{|\cdot|}\right) - 2\alpha\log\alpha\ \delta. \tag{114}$$

Let us split the potential $v_C := v_C^B$ of h_C as

$$v_C = \log B\ v_\delta + v_Q,$$

where v_δ is defined in (17) and

$$v_Q = -Z\sum_j \left(\mathrm{Pf}\ \frac{1}{|z_j|} + C_j^n\delta(z_j)\right) + \sum_{j<k}\left(\mathrm{Pf}\ \frac{1}{|z_j - z_k|} + C_{jk}^e\delta(z_j - z_k)\right),$$

the (pseudo-) Coulombic part. If $\simeq$ denotes unitary equivalence with respect to the dilation ρ_α (on $\mathbb{R}^N$), then in view of our choice of α

$$\begin{aligned} h_C - \mu &\simeq \alpha^2\left(h_{00} + \frac{\log B - 2\log\alpha}{\alpha}v_\delta + \frac{1}{\alpha}v_Q - \alpha^{-2}\mu\right) = \alpha^2(h_{00} + 2v_\delta + \frac{1}{\alpha}v_Q - \alpha^{-2}\mu) \\ &\ge \alpha^2\left(\frac{1}{2}h_{00} - b - \alpha^{-2}\mu\right) \end{aligned}$$

form some $b > 0$ depending only on Z, N and $\mathbb{M}$, since by lemma 4.9 we know that $2v_\delta + \frac{1}{\alpha}v_Q$ is h_{00} form bounded with relative bound 0. Recall that $B \ge e$ implies $\alpha \ge 1$. Choosing $\mu = \mu_C := -\alpha^2(\frac{1}{2} + b) =: -\alpha^2\nu_c$ will insure that $(h_{00} + \alpha^2)r_C(\mu)(h_{00} + \alpha^2)^{1/2} \le 2$ which is what we want to prove. A more careful argument, which we will skip, will yield an explicit b. QED

We continue with the proof of Theorem 6.1. By (113) and (112) with $\mu = \mu_C$, we find that

$$\|K_C(\xi)\| \le \frac{2C_{(72)}}{B^{1/4}\sqrt{\alpha}} \max\left\{\frac{\nu_C\alpha^2}{d_C(\xi)}, 1\right\} \le \frac{1}{2},$$

if both

$$d_C(\xi) \ge 4C_{(72)}\nu_C\ \frac{\alpha^{3/2}}{B^{1/4}} =: C_C'\ \frac{\alpha^{3/2}}{B^{1/4}},$$

and $\alpha^{1/2}B^{1/4} \ge 4C_{(72)}$ which, since $B \mapsto \alpha(B)^2B$ is increasing, is equivalent to $B \ge 4^3C_{(72)}^4\alpha(4C_{(72)}^2)^{-2}$. Since we also need $B \ge e$ we put

$$B_C' := \max\left\{\frac{4^3C_{(72)}^4}{4\alpha\left(C_{(72)}^2\right)^2}, e\right\}.$$

This fixes our constants C_C' and B_C', and also implies that $\|r_{\text{eff}}(\xi)\| \le 2\|r_C(\xi)\|$, by the resolvent formula.

To show that (109) implies that $\|r_C(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$, we repeat the argument with r_C and r_{eff} interchanged: by the resolvent formula,

$$r_C(\xi) = r_{\text{eff}}(\xi)^{1/2}\left(1 + \widetilde{K}_C(\xi)\right)^{-1}|r_{\text{eff}}(\xi)|^{1/2},$$

with

$$\widetilde{K}_C(\xi) := |r_{\text{eff}}(\xi)|^{1/2}(h_C - h_{\text{eff}})r_{\text{eff}}(\xi)^{1/2},$$

so that, using lemmas 4.5, 5.2 and 5.3, we arrive at

$$\|\widetilde{K}_C(\xi)\| \leq \frac{2C_{(72)}}{B^{1/4}\sqrt{\alpha}}\max\left\{\frac{|\mu_{\text{eff}}|}{d_{\text{eff}}(\xi)}, 1\right\} \leq \frac{2C_{(72)}}{B^{1/4}\sqrt{\alpha}}\max\left\{\frac{2|\mu_{\text{eff}}|}{d_C(\xi)}, 1\right\},$$

if $d_C(\xi) \geq C'_C\alpha^{3/2}B^{-1/4}$, by the first part. We therefore conclude that $\|\widetilde{K}_C(\xi)\| \leq 1/2$, and hence $\|r_C(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$, if both $\sqrt{\alpha}B^{1/4} \geq 4C_{(72)}$, which will be satisfied if $B \geq B'_C$ defined above, and if

$$\frac{4|\mu_{\text{eff}}|C_{(72)}}{\sqrt{\alpha}B^{1/4}d_C(\xi)} \leq \frac{1}{2}.$$

The latter inequality is equivalent to

$$d_C(\xi) \geq 4\left(\frac{N}{2\varepsilon_{\text{eff}}^2} + 1\right) C_{(72)}\,\frac{\alpha^{3/2}}{B^{1/4}},$$

which yields condition (109) .

Finally, if ξ satisfies (110), then

$$\begin{aligned}
\|r_{\text{eff}}(\xi) - r_C(\xi)\| &\leq \frac{4C_{(72)}}{\alpha^{1/2}B^{1/4}d_C(\xi)}\max\left\{\frac{\nu_C\alpha^2}{d_C(\xi)}, 1\right\} \\
&\leq 4\nu_C C_{(72)}\frac{\alpha^{3/2}}{d_C(\xi)^2B^{1/4}} \\
&= C'_C\frac{\alpha^{3/2}}{d_C(\xi)^2B^{1/4}},
\end{aligned}$$

where we used that $d_C(\xi) \leq \alpha^2/2 \leq \nu_C\alpha^2$. This finishes the proof of Theorem 6.1. QED

Proof of Theorem 1.3. We define

$$B_C := \max\{B_{\text{eff}}, B'_C\}, \tag{115}$$

and

$$c_C := \max\{C''_c, 2c_{\text{eff}}\alpha(B_C)^{-1/2}B_C^{-1/4}\}. \tag{116}$$

Suppose that $B \geq B_C$, and that

$$c_C\alpha^{3/2}B^{-1/4} \leq d_C(\xi) \leq \frac{1}{4}\alpha^2.$$

By Theorem 6.1, $d_{\text{eff}}(\xi) \leq 2d_C(\xi) \leq \alpha^2/2$. By the same theorem, and by (116),

$$d_{\text{eff}}(\xi) \geq \frac{1}{2}d_C(\xi) \geq c_{\text{eff}}\alpha(B_C)^{-1/2}B_C^{-1/4}\alpha^{\frac{3}{2}}B^{-1/4} \geq c_{\text{eff}}\alpha(B)B^{-1/2},$$

since $\alpha(B)/\sqrt{B}$ is a decreasing function of $B > 0$. The conditions of Theorem 1.1 are therefore met. Since the condition (110) of Theorem 6.1 is clearly also satisfied, we conclude that the difference of the resolvents (15) can be estimated by

$$C_{\text{eff}}\frac{\alpha^2}{d_{\text{eff}}(\xi)^2\sqrt{B}} + C'_C\frac{\alpha^{3/2}}{d_C(\xi)^2B^{1/4}} \leq C_c\frac{\alpha^{3/2}}{d_C(\xi)^2B^{1/4}},$$

for $B \geq B_C$, with

$$C_C := 4C_{\text{eff}} \frac{\sqrt{\alpha(B_C)}}{B_C^{1/4}} + C'_C, \tag{117}$$

where we used that $\alpha(B)/\sqrt{B}$ is decreasing. QED

7. **Proof of Theorem 1.5**

This will be done by closely following the strategy of section 6. First, we compare the resolvents $r_{\text{eff}}(\xi)$ of h_{eff} and $r_\delta(\xi) := (h_\delta - \xi)^{-1}$ of $h_\delta := h_\delta^{B,\mathbb{M}}$. Recall, that $d_\delta(\xi) := \text{dist}\,(\xi, \sigma(h_\delta))$.

Theorem 7.1. *Let $\alpha = \alpha(B)$ be defined by (6). There exist (computable) constants $B'_\delta, C'_\delta \geq 0$, only depending on Z, N and $\mathbb{M}$, such that for all $B \geq B'_\delta$ and all real ξ satisfying*

$$d_\delta(\xi) \geq C'_\delta\, \alpha, \tag{118}$$

we have that $\xi \in \rho(h_{\text{eff}})$, with $\|r_{\text{eff}}(\xi)\| \leq 2\|r_\delta(\xi)\|$. In addition, letting

$$C''_\delta = \max\{C'_\delta, 4C_{(119)}\left(\frac{N}{2\varepsilon_{\text{eff}}^2} + 1\right)\} \quad \text{with} \quad C_{(119)} := \left(NZ + \frac{N(N-1)}{2}\right) C_{(85)}, \tag{119}$$

and ε_{eff} given by (96), then if

$$d_\delta(\xi) \geq C''_\delta\, \alpha, \tag{120}$$

we also have that $\|r_\delta(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$. Finally, if

$$C'_\delta\, \alpha \leq d_\delta(\xi) \leq \frac{1}{2}\alpha^2, \tag{121}$$

then

$$\|r_{\text{eff}}(\xi) - r_\delta(\xi)\| \leq C'_\delta \frac{\alpha}{d_\delta(\xi)^2}. \tag{122}$$

Proof.
As in the proof of Theorem 5.1, 6.1 one shows that

$$\|r_{\text{eff}}(\xi) - r_\delta(\xi)\| \leq \frac{1}{d_\delta(\xi)}\ \frac{\|K_\delta(\xi)\|}{1 - \|K_\delta(\xi)\|},$$

where $K_\delta := |r_\delta(\xi)|^{1/2}(h_{\text{eff}} - h_\delta) r_\delta(\xi)^{1/2}$. Using lemma 4.8 with $c = 2$, lemma 4.6, the triangle inequality, and similar comparison arguments as in the proofs of Theorems 5.1 and 6.1, we easily find that, for any $\mu < \inf\sigma(h_\delta)$,

$$\|K_\delta(\xi)\| \leq \frac{C_{(119)}}{\alpha}\ \max\left\{\frac{|\mu|}{d_\delta(\xi)}, 1\right\}\ \|r_\delta(\mu)^{1/2}(h_{00} + \alpha^2)^{1/2}\|^2, \tag{123}$$

where $C_{(119)}$ was defined above. We then use the following analogue of lemmas 5.3 and 6.2.

Lemma 7.2. *Let $\nu_\delta = 1/2 + 4NZ^2$, $\mu_\delta := -\nu_\delta\alpha^2$ and let α be defined by (6). Then $\mu_\delta < \inf\sigma(h_\delta)$, and*

$$\|r_\delta(\mu_\delta)^{1/2}(h_{00} + \alpha^2)^{1/2}\|^2 \leq 2. \tag{124}$$

Proof. We will use , as before, the scaling $z \mapsto z/\alpha$. We get, with the help of lemma 4.9,

$$\begin{aligned} h_\delta - \mu &\simeq \alpha^2(h_{00} - 2v_\delta - \alpha^{-2}\mu) \geq \alpha^2(h_{00} - 2Z\sum_{j=1}^{N}\delta(z_j) - \alpha^{-2}\mu) \\ &\geq \alpha^2(h_{00}(1-2Z\varepsilon) - N\varepsilon^{-1}Z - \alpha^{-2}\mu) \qquad (\varepsilon > 0) \\ &= \frac{1}{2}\alpha^2(h_{00}+1), \end{aligned}$$

if we choose $\varepsilon = \varepsilon_\delta := 1/(4Z)$ and $\mu = \mu_\delta := -\alpha^2(\frac{1}{2} + 4NZ^2)$. QED

We continue with the proof of Theorem 7.1. By (123) and (124) with $\mu = \mu_\delta$, we find that

$$\|K_\delta(\xi)\| \leq \frac{2C_{(119)}}{\alpha} \max\left\{\frac{\nu_\delta\alpha^2}{d_\delta(\xi)}, 1\right\} \leq \frac{1}{2},$$

if both

$$d_\delta(\xi) \geq 4C_{(119)}\nu_\delta\ \alpha =: C'_\delta\ \alpha,$$

and $\alpha \geq 4C_{(119)}$ which, since $B \mapsto \alpha(B)$ is increasing, is equivalent to

$$B \geq B'_\delta := 16C^2_{(119)}e^{8C_{(119)}}.$$

This fixes our constants C'_δ and B'_δ, and also implies that $\|r_{\text{eff}}(\xi)\| \leq 2\|r_\delta(\xi)\|$, by the resolvent formula. To show that (120) implies that $\|r_\delta(\xi)\| \leq 2\|r_{\text{eff}}(\xi)\|$, we repeat the argument with r_δ and r_{eff} interchanged: by the resolvent formula,

$$r_\delta(\xi) = r_{\text{eff}}(\xi)^{1/2}\left(1 + \widetilde{K}_\delta(\xi)\right)^{-1}|r_{\text{eff}}(\xi)|^{1/2},$$

with

$$\widetilde{K}_\delta(\xi) := |r_{\text{eff}}(\xi)|^{1/2}(h_\delta - h_{\text{eff}})r_{\text{eff}}(\xi)^{1/2},$$

so that, using lemmas 4.8, 5.2 and 5.3, we arrive at

$$\|\widetilde{K}_\delta(\xi)\| \leq 2\max\left\{\frac{|\mu_{\text{eff}}|}{d_{\text{eff}}(\xi)}, 1\right\}\frac{C_{(119)}}{\alpha} \leq 2\max\left\{\frac{2|\mu_{\text{eff}}|}{d_\delta(\xi)}, 1\right\}\frac{C_{(119)}}{\alpha},$$

if $d_\delta(\xi) \geq C'_\delta\alpha$, by the first part. We therefore conclude that $\|\widetilde{K}_\delta(\xi)\| \leq 1/2$ if both $\alpha \geq 4C_{(119)}$, which is satisfied since $B \geq B'_\delta$ defined above, and if

$$\frac{4|\mu_{\text{eff}}|C_{(119)}}{\alpha d_\delta(\xi)} \leq \frac{1}{2}.$$

The latter inequality is equivalent to

$$d_\delta(\xi) \geq 4\left(\frac{N}{2\varepsilon^2_{\text{eff}}} + 1\right)\ C_{(119)}\ \alpha,$$

which yields condition (120) . Finally, if ξ satisfies (121), then

$$\begin{aligned} \|r_{\text{eff}}(\xi) - r_\delta(\xi)\| &\leq \frac{4C_{(119)}}{\alpha d_\delta(\xi)}\max\left\{\frac{\nu_\delta\alpha^2}{d_\delta(\xi)}, 1\right\} \\ &= 4\nu_\delta C_{(119)}\frac{\alpha}{d_\delta(\xi)^2} = C'_\delta\frac{\alpha}{d_\delta(\xi)^2}, \end{aligned}$$

where we used that $d_\delta(\xi) \leq \alpha^2/2 \leq \nu_\delta\alpha^2$. This finishes the proof of Theorem 7.1. QED

Proof of Theorem 1.5. We want to realize the assumptions of Theorem 1.1 and 7.1 with conditions on B and d_δ. We define

$$B_\delta := \max\{B_{\text{eff}}, B'_\delta\},$$

and

$$c_\delta := \max\{C''_\delta, 2c_{\rm eff} B_\delta^{-1/2}\}. \tag{125}$$

Clearly $B \geq B_\delta$ and $c_\delta \alpha \leq d_\delta(\xi) \leq \alpha^2/4$ will do, for under these conditions, using Theorems 1.1 and 7.1, the left hand side of (20) can be estimated by

$$\frac{C_{\rm eff}\alpha^2}{d^2_{\rm eff}(\xi)\sqrt{B}} + \frac{C'_\delta \alpha}{d^2_\delta(\xi)} \leq \left(\frac{4C_{\rm eff}\alpha(B_\delta)}{\sqrt{B_\delta}} + C'_\delta\right)\frac{\alpha}{d_\delta(\xi)^2} =: C_\delta \frac{\alpha}{d_\delta(\xi)^2}$$

since we know that both $2d_{\rm eff}(\xi) \geq d_\delta(\xi)$ x

and $2d_\delta(\xi) \geq d_{\rm eff}(\xi)$, by Theorem 7.1, and since $\alpha(B)/\sqrt{B}$ is a decreasing function of $B > 0$. QED

8. **The fermionic case**

We first prove Theorem 1.6. This is simply done by repeating the proofs of Theorems 1.1, 1.3 and 1.5 for the 'fermionized' operators, that is, for the operators sandwiched between P^{AS}. We have to check that the main ingredients of these proofs remain valid. First of all, corollary 3.3 is used to compare the resolvents of $H^{B,\mathbb{M}}_{\rm f}$ and $H^{B,\mathbb{M}}_{\rm eff,f} + \mathcal{W}_f$. Next, lemma 4.7 remains valid for $\mathcal{W}_f := P^{AS}\mathcal{W}P^{AS}$, with the operator norm being the one on $P^{AS}\left(\text{Ran } \Pi^{B,\mathbb{M}}_{\rm eff}\right)$, since P^{AS} commutes with h_{00} on Ran $\Pi^{B,\mathbb{M}}_{\rm eff} = L^2(\mathbb{R}^N, F^B_{\mathbb{M}})$ (as we will explicitly see below, P^{AS} not only mixes the coordinates of $\mathbb{R}^N$, but also the different components with respect to the natural basis of $F^B_{\mathbb{M}}$; however, h_{00} acts in a scalar way). We then repeat the proof of Theorem 1.1 in section 5, replacing $d_{\rm eff}$ everywhere by $d_{\rm eff,f}$. Similar remarks apply to the proofs of Theorems 1.3 and 1.5. QED

We next turn to Theorem 1.8. The parameter B here plays a non-essential rôle, and we will simply drop it, writing X_m, χ_m, $F_{\mathbb{M}}$ for X^B_m, χ^B_m, $F^B_{\mathbb{M}}$, etc. We start by analyzing the subspace of anti-symmetric wave functions in the range of $\Pi_{\rm eff} := \Pi^B_{\rm eff}$. Recall that

$$\Sigma(\mathbb{M}) = \{m = (m_1, \cdots, m_N) : m_j \geq 0, m_1 + \cdots + m_N = \mathbb{M}\}.$$

The permutation group S_N acts on $\Sigma(\mathbb{M})$ by $\sigma\cdot(m_1, \cdots, m_N) = (m_{\sigma(1)}, \cdots, m_{\sigma(N)})$ and $\Sigma(\mathbb{M})$ can therefore be written as a disjoint union of orbits of S_N :

$$\Sigma(\mathbb{M}) = \bigcup_{\overline{m}\in\mathcal{M}} S_N \cdot \overline{m},$$

$\mathcal{M} \subset \Sigma(\mathbb{M})$ being a set of representatives of ${}_{S_N}\backslash\, \Sigma(\mathbb{M})$. If we let

$$V_{\overline{m}} = \text{Span } \{X_{\sigma\cdot\overline{m}} : \sigma \in S_N\}, \tag{126}$$

then, recalling that $F_{\mathbb{M}} = \text{Span } \{X_m : m \in \Sigma(\mathbb{M})\}$, we have the orthogonal decomposition

$$F_{\mathbb{M}} = \bigoplus_{\overline{m}\in\mathcal{M}} F_{\overline{m}}.$$

From this it follows that

$$\Pi^{1,\mathbb{M}}_{\rm eff}\left(L^2(\mathbb{R}^{3N}) \otimes \mathbb{C}^{2N}\right) = L^2\left(\mathbb{R}^N\right) \otimes F_{\mathbb{M}} = \bigoplus_{\overline{m}\in\mathcal{M}} L^2(\mathbb{R}^N) \otimes F_{\overline{m}}.$$

Since P^{AS} leaves each $L^2(\mathbb{R}^N) \otimes F_{\overline{m}}$ invariant, it suffices to analyze the subspace of anti-symmetric wave functions in each of the latter. We therefore fix an $\overline{m} \in \mathcal{M}$ and let

$$G_{\overline{m}} = \{\sigma \in S_N : \sigma \cdot \overline{m} = \overline{m}\}, \tag{127}$$

the stabilizer of $\overline{m}$. Choose representatives $\sigma_1, \cdots, \sigma_K$, $K = K(\overline{m})$, for the right equivalence classes of $G_{\overline{m}}$ in S_N : $S_N/G_{\overline{m}} = \{\sigma_1 G_{\overline{m}}, \cdots, \sigma_K G_{\overline{m}}\}$ with $\sigma_i G_{\overline{m}} \cap \sigma_j G_{\overline{m}} = \emptyset$ if $i \neq j$. Then $X_{\sigma_1 \cdot \overline{m}}, \cdots, X_{\sigma_K \cdot \overline{m}}$ constitutes an orthonormal basis for $F_{\overline{m}}$, and each element $\psi = \psi(x, y, z)$ of $L^2(\mathbb{R}^N) \otimes F_{\overline{m}}$ can be uniquely written as:

$$\psi = \sum_{j=1}^{K} a_j X_{\sigma_j \cdot \overline{m}} = \sum_{j=1}^{K} a_j(z) X_{\sigma_j \cdot \overline{m}}(x, y), \tag{128}$$

for suitable $a_j = a_j(z) \in L^2(\mathbb{R}^N)$. For any such ψ and $r = (x, y, z) \in \mathbb{R}^{3N} = (\mathbb{R}^N)^3$,

$$\begin{aligned} \psi(\tau \cdot r) &= \sum_j a_j(\tau \cdot z) X_{\sigma_j \cdot \overline{m}}(\tau \cdot x, \tau \cdot y) \\ &= \sum_j a_j(\tau \cdot z) X_{(\tau^{-1}\sigma_j)\cdot \overline{m}}(x, y), \end{aligned}$$

where $\tau \cdot r = (r_{\tau(1)}, \cdots, r_{\tau(N)})$, and similarly for $\tau \cdot x$, $\tau \cdot y$ and $\tau \cdot z$, and where we used that

$$X_m(\tau \cdot x, \tau \cdot y) = \prod_j \chi_{m_j}(x_{\tau(j)}, y_{\tau(j)}) = X_{\tau^{-1} \cdot m}(x, y).$$

It follows that $\psi \in L^2(\mathbb{R}^N) \otimes F_{\overline{m}}$ is anti-symmetric iff, for any $\tau \in S_N$,

$$\sum_j a_j(\tau \cdot z) X_{(\tau^{-1}\sigma_j)\cdot \overline{m}} = (-1)^\tau \sum_j a_j(z) X_{\sigma_j \cdot \overline{m}}.$$

This is equivalent to the statement that ψ is antisymmetric iff

$$\sum_j a_j(\tau \cdot z) X_{(\tau\sigma_j)\cdot \overline{m}} = (-1)^\tau \sum_j a_j(z) X_{\sigma_j \cdot \overline{m}}, \tag{129}$$

since the two statements are equivalent when τ is a transposition, and these generate S_N. The version (129), with no τ^{-1}, will be more convenient to work with. We next observe that the map $\sigma_j \to \tau\sigma_j$ gives rise to a a permutation $\rho(\tau)$ of S_K:

Lemma 8.1. For any $\overline{m} \in \mathcal{M}$ the map $\rho = \rho_{\overline{m}} : S_N \to S_K$, $K = K(\overline{m})$, such that

$$\rho(\tau)(i) = j \Leftrightarrow \tau\sigma_i \in \sigma_j G_{\overline{m}}$$

is a well defined homomorphism.

In other words, $\rho(\tau)$ is characterized by:

$$\tau\sigma_i \in \sigma_{\rho(\tau)(i)} G_{\overline{m}}.$$

One easily verifies that $\rho(\tau)$ is indeed a permutation of $\{1, \cdots, K\}$, and that ρ is an homomorphism of S_N into S_K.

With this notation, the left hand side of (129) reads :

$$\sum_j a_j(\tau \cdot z) X_{\sigma_{\rho(\tau)(j)} \cdot \overline{m}},$$

and on replacing j by $\rho(\tau)^{-1}(j)$ and using the fact that the $X_{\sigma_j \cdot \overline{m}}$ form a basis of $F_{\overline{m}}$, we find that ψ is anti-symmetric iff, for all j and all $\tau \in S_N$,

$$a_j(z) = (-1)^\tau a_{\rho(\tau^{-1})(j)}(\tau \cdot z). \tag{130}$$

If we successively replace τ by τ^{-1} and z by $\tau \cdot z$, this becomes

$$a_j(\tau \cdot z) = (-1)^\tau a_{\rho(\tau)(j)}(z), \tag{131}$$

which implies that all a_j are uniquely determined by any one of them, a_1, say, which we let, by definition, correspond to $\sigma_1 = e$, the unity element of S_N. More

explicitly, since $\rho(\sigma_j)(1) = j$ (for $\sigma_j\sigma_1 = \sigma_j \in \sigma_j G_{\overline{m}} = G_{\rho(\sigma_j)(1)\cdot\overline{m}}$, by definition of ρ), equation (130) with $\tau = \sigma_j$ implies the important relation

$$a_j(z) = (-1)^{\sigma_j} a_1(\sigma_j \cdot z). \tag{132}$$

In particular, $P^{AS}(L^2(\mathbb{R}^N) \otimes F_{\overline{m}})$ can be identified with a subspace of $L^2(\mathbb{R}^N)$, by sending ψ to a_1 (see (133)). We now analyze the symmetry properties of a_1 imposed by the anti-symmetry of ψ.

Lemma 8.2. *Let H be the subgroup of S_N generated by the set $\{\tau\sigma^{-1}_{\rho(\tau)(1)} : \tau \in S_N\}$. Then, for all $\sigma \in H$, $a_1(\sigma \cdot z) = (-1)^\sigma a_1(z)$, and these are the only symmetry-conditions which the anti-symmetry of ψ imposes on a_1.*

Proof. Let $\tau \in S_N$ be arbitrary. Then by (131), $a_1(\tau \cdot z) = (-1)^\tau a_{\rho(\tau)(1)}(z)$ which, by (132), equals $(-1)^\tau(-1)^{\sigma_{\rho(\tau)(1)}} a_1(\sigma_{\rho(\tau)(1)} \cdot z)$. Therefore

$$a_1(\tau\sigma^{-1}_{\rho(\tau)(1)} \cdot z) = (-1)^{\tau\sigma^{-1}_{\rho(\tau)(1)}} a_1(z),$$

whence the lemma. QED

Lemma 8.3. *The group H of lemma 8.2 is generated by the union of all stabilizers $G_{\sigma_j\cdot\overline{m}}$ of $\sigma_j \cdot \overline{m}$, $1 \leq j \leq K$.*

Proof. Let $\tau \in S_N$. Then $\tau \in \sigma_j G_{\overline{m}}$, for some j. Since $\tau\sigma_1 = \tau e \in \sigma_j G_{\overline{m}}$, we have that $\rho(\tau)(1) = j$, and therefore $\tau\sigma^{-1}_{\rho(\tau)(1)} \in \sigma_j G_{\overline{m}} \sigma_j^{-1} = G_{\sigma_j\cdot\overline{m}}$.

Conversely, if $\sigma \in G_{\sigma_j\cdot\overline{m}}$, then $\sigma = \sigma_j\sigma'\sigma_j^{-1}$, for some $\sigma' \in G_{\overline{m}}$. Put $\tau = \sigma_j\sigma'$. Then $\rho(\tau)(1) = j$, since $\sigma_j\sigma'\sigma_1 \in \sigma_j G_{\overline{m}}$, and therefore $\sigma = \tau\sigma_j^{-1} = \tau\sigma^{-1}_{\rho(\tau)(1)}$ is a generator of H. We conclude that the set of generators of H equals $\cup_j G_{\sigma_j\cdot\overline{m}}$, which proves the lemma. QED

Lemma 8.4. *Let H be the subgroup from lemma 8.2. If $G_{\overline{m}} = \{e\}$, then $H = \{e\}$, while if $G_{\overline{m}} \neq \{e\}$, then $H = S_N$.*

Proof. It is obvious, from lemma 8.3, that if $G_{\overline{m}} = \{e\}$, then $H = \{e\}$, and a_1 does not have to satisfy any symmetry-conditions with respect to the action of S_N, by lemma 8.2.

Now suppose that $G_{\overline{m}}$ is non-trivial. Then there exist two indices i and j such that $\overline{m}_i = \overline{m}_j$. We can suppose, without loss of generality, that $i = 1$ and $j = 2$. In that case, the transposition (12) is in $G_{\overline{m}}$, and therefore $(\sigma(1), \sigma(2)) = \sigma(12)\sigma^{-1} \in H$, for all $\sigma \in S_N$, by lemma 8.3 again. But then all transpositions will be in H, which clearly implies that $H = S_N$. QED

Define a linear mapping

$$U_{\overline{m}} : P^{AS}\left(L^2(\mathbb{R}^N) \otimes F_{\overline{m}}\right) \to L^2(\mathbb{R}^N), \tag{133}$$

by

$$U_{\overline{m}}(\psi)(z) = \sqrt{K}(\psi(\cdot, z), X_{\overline{m}})_{L^2(\mathbb{R}^{2N})},$$

where we recall that $K = K(\overline{m}) = \#(S_N/G_{\overline{m}})$. If ψ is given by (128), then $U_{\overline{m}}(\psi)(z) = \sqrt{K} a_1(z)$. By (131), $\|U_{\overline{m}}\psi\|^2 = \|\psi\|^2$, so that $U_{\overline{m}}$ is unitary and therefore injective. If $G_{\overline{m}}$ is non-trivial, then $H = S_N$, and the image of $U_{\overline{m}}$ is contained in the space $L^2_{AS}(\mathbb{R}^N)$ of anti-symmetric wave-functions on $\mathbb{R}^N$, by lemmas 8.2 and 8.4. Since the only symmetries of a_1 are those imposed by H, it follows that $U_{\overline{m}}$ is surjective onto $L^2_{AS}(\mathbb{R}^N)$. Similarly, if $G_{\overline{m}} = \{e\}$, then $H = \{e\}$, and the image of $U_{\overline{m}}$ is $L^2(\mathbb{R}^N)$: indeed, if $a_1 = a_1(z)$ is arbitrary, then

$$\psi_{a_1} := \sum_{\sigma\in S_N} a_1(\sigma \cdot z) X_{\sigma\cdot\overline{m}}(y, z)$$

is an anti-symmetric element of $L^2(\mathbb{R}^N) \otimes F_{\mathbb{M}}$ such that $U_{\overline{m}}(\psi_{a_1}) = a_1$.

Proof of Theorem 1.8. Recall that

$$\mathcal{M}_1 = \{\overline{m} \in \mathcal{M} : G_{\overline{m}} = \{e\}\ \}, \quad \mathcal{M}_2 = \{\overline{m} \in \mathcal{M} : G_{\overline{m}} \neq \{e\}\ \},$$

and define

$$U_{\mathbb{M}} =: U_{\mathbb{M}}^B : P^{AS}\left(L^2(\mathbb{R}^N \otimes F_{\mathbb{M}})\right) \ \to \sum_{\overline{m}\in\mathcal{M}_1}^{\oplus} L^2(\mathbb{R}^N) \oplus \sum_{\overline{m}\in\mathcal{M}_2}^{\oplus} L^2_{AS}(\mathbb{R}^N), \tag{134}$$

by $U_{\mathbb{M}} := \oplus_{\overline{m}\in\mathcal{M}} U_{\overline{m}}$. Then we have shown that $U_{\mathbb{M}}$ is a surjective isometry. The intertwining formula of $h^{\mathbb{M}}_{\delta,\mathrm{f}}$ with $U_{\mathbb{M}}$ being obvious, this proves Theorem 1.8. QED

Contrary to $h^{\mathbb{M}}_{\delta,\mathrm{f}}$, the operator $U_{\mathbb{M}} h^{\mathbb{M}}_{C,\mathrm{f}} U^*_{\mathbb{M}}$ will in general not act diagonally anymore on the range of $U_{\mathbb{M}}$, but will contain terms which couple anti-symmetric and boltzonic components in (134), that is, components in $L^2_{AS}(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$. The potentially problematic terms in $h^{\mathbb{M}}_{C,\mathrm{f}}$ are $U_{\mathbb{M}} C^{n,\mathbb{M}}_{\mathrm{av}:1} U^*_{\mathbb{M}}$ and $U_{\mathbb{M}} C^{e,\mathbb{M}}_{\mathrm{av}:2} U^*_{\mathbb{M}}$ (still dropping the B from our notations). The first one is easily seen to act diagonally on the right hand side of (134): recall that

$$C^{n,\mathbb{M}}_{\mathrm{av}:1} = -\Pi^1_{\mathrm{eff}} \left(\frac{1}{N} \sum_{j=1}^N \log(\frac{1}{4}\rho_j^2) \right) \Pi^1_{\mathrm{eff}},$$

and identify this with a Hermitian operator on $F_{\mathbb{M}} = \mathrm{Span}\ \{X_m : m \in \Sigma(\mathbb{M})\}$ (it acts as a multiplication operator on $L^2(\mathbb{R}^N, F_{\mathbb{M}})$). The matrix of $C^{n,\mathbb{M}}_{\mathrm{av}:1}$ in the basis X_m, $m \in \Sigma(\mathbb{M})$, is easily seen to be diagonal. Moreover,

$$\begin{aligned} \langle X_{\sigma\cdot m} | C^{n,\mathbb{M}}_{\mathrm{av}:1} | X_{\sigma \cdot m}\rangle &= -\int_{\mathbb{R}^{2N}} |X_m(\sigma^{-1}\cdot(x,y)|^2 \log\frac{1}{4}\left(\Pi_{j=1}^N \rho_j^2\right)^{1/N}\ dxdy \\ &= \langle X_m | C^{n,\mathbb{M}}_{\mathrm{av}:1} | X_m \rangle. \end{aligned}$$

Hence, taking $B = 1$, for simplicity, $C^{n,\mathbb{M}}_{\mathrm{av}:1}$ will act on $F_{\overline{m}}$ as scalar multiplication by

$$\begin{aligned} \langle X_{\overline{m}} C^{n,\mathbb{M}}_{\mathrm{av}:1} | X_{\overline{m}} \rangle &= -\frac{1}{N}\sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(\log\tfrac{\rho_j^2}{4}\right)\ \Pi_{\nu=1}^N |\chi^1_{\overline{m}_\nu}|^2(\rho_\nu)\ dx_\nu dy_\nu \\ &= -\frac{1}{N}\sum_{j=1}^N \int_{\mathbb{R}^2} \left(\log\tfrac{\rho^2}{4}\right)\ |\chi^1_{\overline{m}_j}|^2(\rho) dxdy \\ &= -\frac{1}{N}\sum_{j=1}^N \left(2^{\overline{m}_j}\overline{m}_j!\right)^{-1} \int_0^\infty \rho^{2\overline{m}_j+1} \log\tfrac{\rho^2}{4}\ e^{-\rho^2/2} d\rho \\ &= \log 2 - \frac{1}{N}\left(\sum_{j=1}^N \psi(\overline{m}_j + 1)\right), \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$, the logarithmic derivative of the Γ-function. Hence $U_{\mathbb{M}} C^{n,\mathbb{M}}_{\mathrm{av}:1} U^*_{\mathbb{M}}$ simply acts diagonally on the right hand side of (134), and more precisely, by scalar multiplication in each component.

The operator $U_{\mathbb{M}} C^{e,\mathbb{M}}_{\mathrm{av}:2} U^*_{\mathbb{M}}$ is more complicated: it is not going to be diagonal in the natural basis and it will in general mix the different components Ran $U_{\mathbb{M}}$, even those with index in $\mathcal{M}_1$ and $\mathcal{M}_2$. This is already the case in the simplest case in which both $\mathcal{M}_1$ and $\mathcal{M}_2$ are non-empty, namely that of two electrons, $N = 2$, and a total angular momentum of $\mathbb{M} = 2$. In that case $\Sigma(\mathbb{M})$ is the union of two orbits under S_2, namely $\{(0,2),(2,0)\}$ and $\{(1,1)\}$ the first having as stabilizer the

identity, and the second having as stabilizer the full group S_2. We can therefore take $\mathcal{M}_1 = \{(0,2)\}$ and $\mathcal{M}_1 = \{(1,1)\}$. We will now compute the matrix element

$$\langle X^1_{(0,2)} | C^{e,2}_{\mathrm{av}:2} | X^1_{(1,1)} \rangle, \tag{135}$$

and simply observe that the result is non-zero. For this computation it is convenient to use complex notation for the lowest Landau functions (25): if we let $\zeta = x + iy$, then (taking $B = 1$ again)

$$\chi^1_m = c_m \zeta^m e^{-|\zeta|^2/4},$$

where $c_m = (2\pi 2^m m!)^{-1/2}$. We then find that (135) equals

$$-c_0 c_2 c_1^2 \int_{\mathbb{C}} \int_{\mathbb{C}} \zeta_1 \zeta_2 \overline{\zeta}_2^2 \, \log\left(\frac{|\zeta_1 - \zeta_2|^2}{4}\right) e^{-(|\zeta_1|^2 + \zeta_2|^2)/2} \, d\zeta_1 d\zeta_2.$$

Making the (by now familiar) change of variables $u = (\zeta_1 + \zeta_2)/\sqrt{2}$, $v = (\zeta_1 - \zeta_2)/\sqrt{2}$, this integral becomes

$$-\frac{c_0 c_2 c_1^2}{4} \int_{\mathbb{C}} \int_{\mathbb{C}} (u^2 - v^2)(\overline{u} - \overline{v})^2 \, \log\left(|v|^2/8)\right) \, e^{-(|u|^2 + |v|^2)/2} \, du dv.$$

Since $(u^2 - v^2)(\overline{u}^2 - 2\overline{uv} + \overline{v}^2) = |u|^4 - |v|^4 + 2i \,\mathrm{Im}\, u^2 \overline{v}^2 + 2\overline{uv}(v^2 - u^2)$, and since, in general,

$$\int_{\mathbb{C}} \int_{\mathbb{C}} u^\alpha \overline{u}^\beta v^\nu \overline{v}^\kappa g(|u|^2, |v|^2) du dv = 0,$$

unless $\alpha = \beta$ and $\nu = \kappa$, we obtain that (135) equals

$$\begin{aligned} & -\frac{c_0 c_2 c_1^2}{4} \int_{\mathbb{C}} \int_{\mathbb{C}} (|u|^4 - |v|^4) \log(|v|^2/8) e^{-(|u|^2 + |v|^2)/4} du dv \\ = \quad & -\frac{c_0 c_2 c_1^2}{4} \left\{ \int_{\mathbb{C}} |u|^4 e^{-|u|^2/2} du \int_{\mathbb{C}} \log(|v|^2/8) e^{-|v|^2/2} dv \right. \\ & \left. - \int_{\mathbb{C}} e^{-|u|^2/2} du \int_{\mathbb{C}} |v|^4 \log(|v|^2/8) e^{-|v|^2/2} dv \right\} = \frac{3}{16\sqrt{2}}, \end{aligned}$$

as can be shown using the Γ-function and its derivative. So (135) is non-zero, and $U_{\mathbb{M}} h^{\mathbb{M}}_{C,\mathrm{f}} U^*_{\mathbb{M}}$ mixes the two sectors.

9. **Concluding remarks**

We finally want to give an idea what these effective Hamiltonians might be good for. Our original motivation for introducing them was for studying the structure of the bottom of spectrum of $H^{B,\mathbb{M}}$, in connection with the maximum ionization problem for atoms in strong magnetic fields; see below. To illustrate how this works, we first consider the comparison with $H^{B,\mathbb{M}}_\delta = h^{B,\mathbb{M}}_\delta \oplus H^{B,\mathbb{M}}_\perp$; cf. Theorem 1.5. Since, by construction, $B_\delta > B_{(36)}$, we know from Theorem 3.1 that $\sigma\left((H^{B,\mathbb{M}}_\perp\right) \subset (0,\infty)$ if $B > B_\delta$. Let $E_\delta := \inf h^B_\delta$, then clearly $E_\delta < 0$.

Since h^B_δ is unitarily equivalent to $\alpha^2(-\frac{1}{2}\Delta_z + 2v_\delta)$ we conclude that whenever E_δ is an isolated eigenvalue, its position as well as its isolation distance is proportional to α^2. That E_δ is an eigenvalue is true for Z large enough when N is fixed; to determine how big Z has to be exactly, relative to N, for this to happen is an open problem (see below). Let us assume henceforth that E_δ is an eigenvalue, which then necessarily is simple. Consequently E_δ is an eigenvalue of $H^{B,\mathbb{M}}_\delta$ with multiplicity $\dim F^B_{\mathbb{M}}$, see (18). Choose two points $\xi_\pm := E_\delta \pm C\alpha^2$ in $\rho(H^{B,\mathbb{M}}_\delta) \cap \rho(H^{B,\mathbb{M}})$, which satisfy (19). This is possible when $B > B_\delta$, see Theorem 1.5. Let Γ be the circle in the complex plane centered at E_δ with radius $C\alpha^2$ and define P and P_δ as the eigenprojections associated to $H^{B,\mathbb{M}}$ and $h^{B,\mathbb{M}}_\delta$, respectively, onto their spectrum inside Γ. To estimate $P - P_\delta$ we need a bound on $R(\xi) - R_\delta(\xi) :=$

$(H^{B,\mathbb{M}}-\xi)^{-1}-(H_\delta^{B,\mathbb{M}}-\xi)^{-1}$ for all $\xi \in \Gamma$. We know already that $\|R(\xi_\pm)-R_\delta(\xi)\| \leq C_\delta \alpha/d_\delta(\xi_\pm)^2 = C_\delta C^{-2}\alpha^{-3}$ by theorem 1.5. To propagate this estimate on all of Γ we use the convenient formula (see [K, IV.(3.10)])

$$\|R(\xi)-R_\delta(\xi)\| \leq \frac{\left\|\frac{H_\delta^{B,\mathbb{M}}-\xi_\pm}{H_\delta^{B,\mathbb{M}}-\xi}\right\|^2 \|R(\xi_\pm)-R_\delta(\xi_\pm)\|}{1-|\xi-\xi_\pm|\left\|\frac{H_\delta^{B,\mathbb{M}}-\xi_\pm}{H_\delta^{B,\mathbb{M}}-\xi}\right\|\|R(\xi_\pm)-R_\delta(\xi)\|} \leq \frac{C_\delta C^{-2}\alpha^{-3}}{1-\sqrt{2}C_\delta C^{-1}\alpha^{-1}}.$$

Then integrating over the contour Γ finally gives $\|P-P_\delta\| = \mathcal{O}(\alpha^{-1})$ as B tends to infinity. This shows that for B large enough these two projections have the same dimension and since they are continuous with respect to B we finally get that for all $B > B_\delta$, $\dim P = \dim P_\delta = \dim F_{\mathbb{M}}^B$.

Our conclusion is therefore that, for sufficiently large B, $H^{B,\mathbb{M}}$ will have a cluster of eigenvalues in the interval $(E_\delta - c_\delta\alpha^2, E_\delta + c_\delta\alpha^2)$ with total multiplicity of $\dim F_{\mathbb{M}}^B$, and apart from this no eigenvalues at a distance $C\alpha^2$ from E_δ, (the allowed B's will depend on C), so that the cluster is separated from the rest of the spectrum of by a distance proportional to α^2. In the particular case when $\dim F_{\mathbb{M}}^B = 1$, i.e. $\mathbb{M} = 0$ or $N = 1$, we get an estimate on the difference of the eigenvectors $\Phi - \Phi_\delta = \mathcal{O}(\alpha^{-1})$ as $B \to \infty$ (in the L^2 norm). Here Φ denotes the eigenvector of $H^{B,\mathbb{M}}$ and $\Phi_\delta(x,y,z) := \varphi_\delta^B(z) X_{m=0}^B(x,y)$ if $\mathbb{M} = 0$, where φ_δ^B is the ground state of h_δ^B. In the case $N = 1$ one has $\varphi_\delta^B(z) = \sqrt{2\alpha Z}\, e^{-2\alpha Z|z|}$.

Consider now the comparison with $h_C^{B,\mathbb{M}}$. The difficulty here is to find the necessary a priori information on the structure of $\sigma(h_C^{B,\mathbb{M}})$. In the case of $N = 1$, using the invariance of $\sigma(h_C^{B,\mathbb{M}})$ under the reflexion $z \mapsto -z$, and the characterisation of the domain of $\sigma(h_C^{B,\mathbb{M}})$ in Appendix A, one sees that the odd spectrum[6] is B-independent and coincides with the spectrum of the hydrogen in the s sector of symmetry. Since the even spectrum intertwines with the odd spectrum and since it is monotonically decreasing with respect to B, cf. (11), it is easy to realize that $\sigma(h_C^{B,\mathbb{M}}) \cap \mathbb{R}_-$, apart from the ground state energy, is made up of clusters of two eigenvalues, the clusters being separated by a distance of order 1 as $B \to \infty$. Thus by using Theorem 1.3 and following a similar strategy as above one can conclude that for $N = 1$, and arbitrarily small ε, $\sigma(H^{B,\mathbb{M}}) \cap (-\infty, -\varepsilon]$ has the same cluster structure for $B > B_\varepsilon$ sufficiently large. Moreover this spectrum deviates from the one of the Coulomb model by at most $c_C \alpha(B)^{\frac{3}{2}} B^{-\frac{1}{4}}$ as $B \to \infty$.

The model operator $h_{\text{eff}}^{B,\mathbb{M}}$ is for the moment of mainly theoretical interest since it does not seem to be solvable even in the one electron case. Notice however that one could solve $h_{\text{eff}}^{B,\mathbb{M}}$ numerically, at least for few electrons and small $\mathbb{M}$, and subsequently use Theorem 1.1 to approximate the true spectrum of $H^{B,\mathbb{M}}$ for large B. Given the non-trivial dimension reduction achieved by theorem 1.1 (from wave-functions of $3N$ variables to ones of N variables, albeit vector-valued) such a procedure would, from a numerical point of view, seem preferable to attacking $H^{B,\mathbb{M}}$ directly.

Whether the simpler models, i.e. the delta and the Coulomb model, are solvable in the N-electron case, $N \geq 2$, is a challenging question in view of applications. We are thinking in particular of the problem of *determining the maximum number N_c of electrons which a clamped nucleus with charge Z can bind when an intense homogeneous magnetic field is applied.* [LSY] has shown that $\liminf N_c/Z \geq 2$ as $Z, B/Z^3 \to \infty$. Very little precise is known for fixed Z and high B. It is conjectured that there should be a B-independent absolute (that is, non-asymptotic) upper

[6]that is, the spectrum of h_C restricted to the odd wave functions

bound of the form $N_c \leq aZ + b$, similar to Lieb's bound $N_c \leq 2Z + 1$ valid when $B = 0$, but this is as yet unproved. Some weaker results are known, of which the best to date is the one of [Sei]; see also [BR] for work on heuristic models related to our $h_{\text{eff}}^{B,\mathbb{M}}$. It is natural to first try to solve the maximal binding question for h_δ^B, or any of our other effective Hamiltonian, and use the approximation theorems of this paper to draw conclusions for $H^{B,\mathbb{M}}$ itself. Some modest progress is possible in this way. It is for example known that the delta model with two electrons is at least numerically solvable, see [Ros], and that this model possesses a unique bound state at the bottom of its spectrum as long as $Z > 0.375$. Therefore using Theorem 1.5 we see that for all $Z > 0.375$ there exists $B_Z \geq 0$ such that for all $B \geq B_Z$ one nucleus with such a charge can bind two electrons. As a consequence, Lieb's bound of $N_c \leq 2Z + 1$ is no longer valid in strong magnetic fields. For general Z, no maximum ionization bound for the δ-model is known as yet.

Let us now briefly turn to the effect of particle symmetry. It follows from Theorems 1.6 and 1.8 that $H_{\text{f}}^{B,\mathbb{M}}$ can be approximated by a direct sum of copies of h_δ^B acting on anti-symmetric $L^2(\mathbb{R}^N)$ plus a direct sum of copies of h_δ^B acting on $L^2(\mathbb{R}^N)$ *without any symmetry condition*. The latter will occur iff $\mathcal{M}_1 \neq \emptyset$, which is the case iff $\mathbb{M} \geq 0 + 1 + \cdots + (N-1) = \frac{1}{2}N(N-1)$. For such $\mathbb{M}$, the ground state energy will be approximately that of boltzonic h_δ^B, which is also the ground state energy of the bosonic h_δ^B, and the same can be shown to be the case for the ground state wave function (assuming there is one), by standard permutation arguments. In fact, Theorem 1.6 plus Theorem 1.8 predict the existence of a cluster of $\#\mathcal{M}_1$ eigenvalues at the bottom of the spectrum of $H_{\text{f}}^{B,\mathbb{M}}$, at a distance of order $\alpha(B)^2$ from the origin as $B \to \infty$.

A further interesting corollary to Theorem 1.8 can be obtained by considering $\mathbb{M}$ as a free parameter. If h_δ^B possesses a ground state, whose energy is isolated in its spectrum, then for sufficiently large B, $H_{\text{f}}^B := P^{\text{AS}} H^B$ will assume its ground state for an $\mathbb{M} \geq \frac{1}{2}N(N-1)$. Stated otherwise, assuming there is a mechanism for transfer of the angular momentum (e.g. emission and absorption of photons), atoms in strong magnetic fields will have an orbital angular momentum in the field direction of at least $\frac{1}{2}N(N-1)$. A natural conjecture is that we have equality here. Notice that this conjecture was shown to be true in the case of $N = 1$ in [AHS], see also [BaSe].

We mention one further application of these effective Hamiltonians. After the location of the spectrum to leading order one can now use regular perturbation theory to compute lower order corrections. We have shown how this can be done in [BeBDP]. This seems definitely more convenient than variational techniques and more familiar than the Birman-Schwinger method used in [AHS] for the one electron case. Continuing for example the above comparison of $H^{B,\mathbb{M}}$ with $h_\delta^{B,\mathbb{M}}$, it is immediate to realize that adding the first order perturbative correction will give an error of order 1. In case $N = 1$ we get that the ground state energy of $H^{B,\mathbb{M}}$ is equal to $\langle h_{\text{eff}}^{B,\mathbb{M}} \Phi_\delta, \Phi_\delta \rangle + \mathcal{O}(1)$ with $\Phi_\delta(x, y, z) = \sqrt{2\alpha Z} e^{-2\alpha Z\, |z|}\, \chi_{\mathbb{M}}^B(x, y)$. This should be compared to Theorem 2.5 of [AHS]. In fact, one can write the ground state energy as a convergent power series, each term of which being of order α^{-k} for k running from -2 to infinity. However, as pointed out in [AHS], in view of the $\log(B)$ behaviour of α this series is of limited value. The situation will be much better with the Coulomb model $h_C^{B,\mathbb{M}}$ since the perturbation series will converge much faster because of the $\alpha^{\frac{3}{2}} B^{-\frac{1}{4}}$ behaviour of the r.h.s. of (15). We hope to tackle this program soon.

The above remarks concentrated on applications of our effective Hamiltonians to the ground state energy of $H^{B,\mathbb{M}}$, but their potential interest is not limited to

that. As the example of h_C shows, other parts of the discrete spectrum of $H^{B,\mathbb{M}}$ will also become amenable to analysis, if this is the case for the effective operator. On a conceptual level, Theorem 1.1 gives a precise mathematical sense to, and justification of, the physicist's attractive heuristic picture of an atom in a strong homogeneous magnetic field as consisting of electrons in their lowest Landau band states interacting through a kind of "residual" electrostatic interaction. Finally let us note that the technics developped in this article are expected to work in other contexts. An interesting example is that of 2-dimensional electronic systems on a cylinder which describe excitons in carbon nanotubes; cf. [CDP].

Appendix A. **A characterization of the operator domain of h_C**

In this appendix we will characterize the operator domain of $h_C = h_C^B$. It will in fact be convenient to consider a slightly more general situation. Let $\mathcal{L} = \{L_\nu, 1 \leq \nu \leq K\}$, be a finite collection of hyperplanes in $\mathbb{R}^N$ (we might more generally consider non-singular C^1-hypersurfaces). Let F be a finite dimensional complex vector space, with Hermitian inner product $(\cdot,\cdot)$ and let A_ν, B_ν be Hermitian operators on F. Let $H^k(\mathbb{R}^N, F)$ be the k-th Sobolev space on $\mathbb{R}^N$, with values in F. Then the following sesquilinear form is well-defined on $H^1(\mathbb{R}^N, F)$ and bounded from below:

$$t_{\mathcal{L}}(u,v) = \frac{1}{2}(\nabla u, \nabla v) + \sum_\nu \left\langle L_\nu^* \mathrm{Pf}\ (|\cdot|^{-1}), (A_\nu u, v)\right\rangle + \left\langle L_\nu^* \delta, (B_\nu u, v)\right\rangle, \tag{136}$$

$\langle \cdot,\cdot\rangle$ denoting the duality between distributions and test functions. We let

$$h_{\mathcal{L}} = -\frac{1}{2}\Delta_z + \sum_\nu A_\nu \mathrm{Pf}\ \left(\frac{1}{L(z)}\right) + B_\nu \delta(L_\nu(z)).$$

be the associated self-adjoint operator, whose existence is guaranteed by the KL^{2}MN-Theorem. Let $\mathcal{R}$ be the set of connected components of $\mathbb{R}^N \setminus \cup_\nu \mathrm{Ker}\ L_\nu$, so that $\mathbb{R}^N \setminus \cup_\nu \mathrm{Ker}\ L_\nu = \cup_{R\in\mathcal{R}} R$ and $R \cap R' = \emptyset$, if $R, R' \in \mathcal{R}, R \neq R'$. Note that, on any of these components R, $L_j^* \mathrm{Pf}(1/|\cdot|)$ simply equals the function $1/|L_j(z)|$. We will identify L_j with an element of $\mathbb{R}^N$, using the Euclidean inner product on $\mathbb{R}^N$; the latter will be denoted by a dot: $z \cdot w$, to distinguish it from the Hermitian inner product (v,w) on F. If we let $H^2(R,F)$ be the F-valued Sobolev space of order 2 on the open subset $R \subset \mathbb{R}^N$, then the domain of $h_{\mathcal{L}}$ can be characterized as follows:

Theorem A.1. *Let $u \in L^2(\mathbb{R}^N, F)$. Then $u \in \mathrm{Dom}(h_{\mathcal{L}})$ iff the following three conditions hold:*

(i) $u \in H^1(\mathbb{R}^N, F)$,

(ii) For each $R \in \mathcal{R}$,

$$-\frac{1}{2}\Delta u + \sum_\nu \frac{1}{|L_\nu(z)|} A_\nu u \in H^2(R, F),$$

(iii) For each j and for each $x \in \{L_j = 0\} \setminus \cup_{k\neq j}\{L_k = 0\}$:

$$L_j \cdot \nabla u\left(x + \varepsilon \frac{L_j}{\|L_j\|}\right) - L_j \cdot \nabla u\left(x - \varepsilon \frac{L_j}{\|L_j\|}\right) \simeq -4\log\varepsilon A_j u(x) + 2B_j u(x) + o(1),$$

as $\varepsilon \to 0$.

Proof. We will use the following characterization (see [K]) of $\mathrm{Dom}(h_{\mathcal{L}})$:

$$u \in \mathrm{Dom}(h_{\mathcal{L}}) \iff \begin{cases} u \in \mathrm{Dom}(t_{\mathcal{L}}) = H^1(\mathbb{R}^N, F), \\ \text{and} \\ |t_{\mathcal{L}}(u,v)| \leq C_u \|v\|^2,\ \forall v \in \mathrm{Dom}(t_{\mathcal{L}}) = H^1(\mathbb{R}^N, F), \end{cases} \tag{137}$$

the norm on the right being the L^2-norm. Here we may, and will, suppose without loss of generality that $v \in C_c^1(\mathbb{R}^N, F)$, the space of compactly supported F-valued C^1-functions.

To analyze (137), we will first establish a convenient integral expression for the pull-backs of δ and of $\mathrm{Pf}\,(1/|\cdot|)$ under a linear map $L : \mathbb{R}^N \to \mathbb{R}$. Using the Euclidean inner product, we can identify L with an element of $\mathbb{R}^N$, which we also denote by L. Using the definition of pull-back (cf. Hörmander [Ho], chapter 6), one easily shows that if $d\sigma_L$ denotes the Euclidean surface measure on $\mathrm{Ker}(L)$, and $\|L\|$ is the Euclidean norm of L, then

$$\langle L^*\delta, \varphi\rangle = \int_{\{L=0\}} \frac{\varphi}{\|L\|} d\sigma_L, \tag{138}$$

and

$$\langle L^*\mathrm{Pf}\,(1/|\cdot|), \varphi\rangle = -\int_{\mathbb{R}^N} \frac{\mathrm{sgn}(L(z)))\log|L(z)|}{\|L\|^2}\, L\cdot\nabla u(z)dz. \tag{139}$$

In fact, since taking pull-backs is coordinate invariant, it suffices to verify these formulas in a orthogonal coordinate system in which $L = (\|L\|, 0, \cdots, 0)$.

We next observe that, for each $R \in \mathcal{R}$, there exists a (unique) function $s_R : \{1, \cdots, K\} \to \{0, 1\}$, such that

$$R = \{z \in \mathbb{R}^N : (-1)^{s_R(j)} L_j(z) \geq 0, j = 1, \cdots, K\},$$

and if $\varepsilon > 0$, we define R_ε by:

$$R_\varepsilon = \{z \in \mathbb{R}^N : (-1)^{s_R(j)} L_j(z) \geq \varepsilon,\ j = 1, \cdots, K\}.$$

Observe that the boundary of R_ε, as well as that of R, are polyhedrae, each of whose faces are contained in one the hypersurfaces $\{(-1)^{s_R(j)}L_j = \varepsilon\}$ and $\{L_j = 0\}$, respectively. On such a face, the outward-pointing normal $n_{R,\varepsilon}$ can be identified with the vector $-(-1)^{s_R(j)}L_j/\|L_j\|$ (translated to the relevant base point on the face, to be precise). Note, that $n_{R,\varepsilon}$ is only defined a.e. on the boundary (with respect to the surface measure). This will not cause difficulties, though.

Suppose now that $u \in \mathrm{Dom}(h_{\mathcal{L}})$ and in particular satisfies the conditions (137). We then have, for any $v \in C_c^1(\mathbb{R}^N, F)$,

$$\begin{aligned} t_{\mathcal{L}}(u,v) &= \lim_{\varepsilon\to 0} \Bigg\{ \frac{1}{2}\sum_{R\in\mathcal{R}} \int_{R_\varepsilon} (\nabla u, \nabla v) dz \\ &- \sum_\nu \sum_R \int_{R_\varepsilon} \frac{\mathrm{sgn}\ (L_\nu)\log|L_\nu|}{\|L_\nu\|^2}\ (L_\nu\cdot\nabla)(A_\nu u, v)\ dz \\ &+ \sum_\nu \int_{\{L_\nu=0\}} \frac{(B_\nu u, u)}{\|L_\nu\|}\ d\sigma_{L_\nu} \Bigg\}. \end{aligned} \tag{140}$$

We want to apply Gauss' divergence Theorem to each of the integrals over R_ε: this is allowed since if $u \in H^1(\mathbb{R}^N, F)$ satisfies (137), then by choosing v compactly supported in R, we see that

$$-\frac{1}{2}\Delta_z u + \sum_\nu A_\nu \left(\frac{1}{L(z)}\right) u \in L^2_{\mathrm{loc}}(R, F),$$

which obviously implies that $\Delta u \in L^2_{\mathrm{loc}}(R, F)$ (since we are staying away from the singularities on the hyperplanes), and hence $u \in H^2_{\mathrm{loc}}(R, F)$. Now

$$\int_{R_\varepsilon} (\nabla u, \nabla v) dz = \int_{R_\varepsilon} (-\Delta u, v) dz + \int_{\partial R_\varepsilon} (n_{R,\varepsilon}\cdot\nabla u, v) d\sigma_{R,\varepsilon}, \tag{141}$$

$d\sigma_{R,\varepsilon}$ being the euclidean surface measure on the boundary. Furthermore,

$$-\|L_\nu\|^{-2}\int_{R_\varepsilon} \mathrm{sgn}\,(L_\nu)\log|L_\nu|\,(L_\nu\cdot\nabla)\,[(A_\nu u,v)]\;dz =$$
$$\int_{R_\varepsilon}\frac{(A_\nu u,v)}{|L_\nu(z)|}dz-\|L_\nu\|^{-2}\int_{\partial R_\varepsilon}\mathrm{sgn}(L_\nu)\;\log|L_\nu|\,(A_\nu u,v)\,(L_\nu\cdot n_{R,\varepsilon})\;d\sigma_{R,\varepsilon}.$$

Using this, the second term in (140) can, after re-arranging, be written as:

$$\begin{aligned}
&\sum_{R\in\mathcal{R}}\int_{R_\varepsilon}\left(\sum_j\frac{(A_ju,v)}{|L_j(z)|}]\right)\;dz\\
+\quad&\sum_j\sum_{\pm}\;\pm\|L_j\|^{-1}\int_{\substack{L_j\,=\,\pm\varepsilon,\\ |L_\nu|\,>\,\varepsilon\text{ if }\nu\neq j}}\mathrm{sgn}(L_j(z))\;\log|L_j(z)|\,(A_ju,v)\\
+\quad&\sum_{j,k:j\neq k}\sum_{\pm}\;\pm\|L_j\|^{-1}\int_{\substack{L_k\,=\,\pm\varepsilon,\\ |L_\nu|\,>\,\varepsilon\text{ if }\nu\neq k}}\mathrm{sgn}(L_j(z))\;\log|L_j(z)|\,(A_ju,v),
\end{aligned}$$

the surface integrals being with respect to the natural surface measures. Since v is compactly supported, the third sum will vanish, in the limit of $\varepsilon\to 0$. This follows from the local integrability of $u\cdot\log|L_j|$ on $\{L_k=0\}$ (if $j\neq k$), which can be seen as follows. Since $u\in H^1$, its restriction $u|_{\{L_k=0\}}$ is in (vector-valued) L^2. On the other hand, $\log|L_j|$, restricted to $\{L_k=0\}$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^{N-1})$, and therefore $u\log|L_j|$, restricted to $\{L_k=0\}$ is locally integrable. Rearranging also the sum over $\mathcal{R}$ of the boundary terms in (141) as a sum of integrals over the various hypersurfaces $\{L_j=\pm\varepsilon\}$, we conclude that for $v\in C^1_c(\mathbb{R}^N,F)$,

$$\begin{aligned}
t_{\mathcal{L}(u,v)}\quad=\quad&\lim_{\varepsilon\to 0}\int_{R_\varepsilon}\left(-\frac{1}{2}\Delta u+\sum_\nu\frac{A_\nu u}{|L_\nu(z)|}\;,v\right)dz\\
+\quad&\sum_j\sum_\pm\;\mp\frac{1}{2}\int_{L_j=\pm\varepsilon,|L_\nu|>\varepsilon(\nu\neq j)}\frac{(L_j\cdot\nabla u,v)}{\|L_j\|}d\sigma_{\{L_j=\pm\varepsilon\}}\\
+\quad&\sum_j 2\log\varepsilon\sum_\pm\int_{L_j=\pm\varepsilon,|L_\nu|>\varepsilon(\nu\neq j)}\frac{(A_ju,v)}{\|L_j\|}d\sigma_{\{L_j=\pm\varepsilon\}}\\
+\quad&\int_{\{L_j=0\}}\frac{(B_ju,v)}{\|L_j\|}d\sigma_{\{L_j=0\}}.
\end{aligned}$$

Since $u\in H^1$ implies that $\|u(\cdot+\varepsilon L_j/\|L_j\|)-u(\cdot)\|_{L^2(\{L_j=0\})}=O(\sqrt{\varepsilon})$, and since $(u,v)\in L^1(\{L_j=0\})$, we can replace the next to last term by

$$2\log\varepsilon\|L_j\|^{-1}\int_{\{L_j=0\}}(A_ju,v)d\sigma_{\{L_j=0\}}+o(1),\quad\varepsilon\to 0.$$

We can, for the same reason, replace v in the second integral by its restriction to $\{L_j=0\}$. Letting $\varepsilon\to 0$, it follows that if u satisfies (137), then it satisfies (i), (ii) and (iii).

Conversely, suppose that u satisfies conditions (i), (ii) and (iii) of the theorem. Then running the argument backwards shows that $|t_{\mathcal{L}}(u,v)|\leq C_u\|v\|^2$, first for all compactly supported v and thence for all $v\in H^1(\mathbb{R}^N,F)$. Hence $u\in\mathrm{Dom}(\mathrm{h}_{\mathcal{L}})$, by (137). QED

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